DUBLIN UNIVERSITY PRESS SERIES.

A SEQUEL TO THE FIRST SIX BOOKS

OF THE

ELEMENTS OF EUCLID,

CONTAINING

AN EASY INTRODUCTION TO MODERN GEOMETRY.

With numerous Examples.

BY

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PREFACE.

I HAVE endeavoured in this Manual to collect and arrange all those elementary Geometrical Propositions not given in Euclid, which a Student will require in his Mathematical Course. The necessity for such a work will be obvions to every person engaged in Mathematical tuition. I have been frequently obliged, when teaching the Higher Mathematics, to interrupt my demonstrations, in order to prove some elementary Propositions on which they depended, but which were not given in any book to which I could refer. The object of the present little Treatise is to supply that want.

The following is the plan of the Work. It is divided into five Chapters, corresponding to Books I., II., III., IV., VI. of Euclid. The Supplements to Books I.–IV. consist of two Sections each, namely, Section I., Additional Propositions; Section II., Exercises. This part will be found to contain original proofs of some of the

most elegant Propositions in Geometry. The Supplement to Book VI. is the most important; it embraces more than half the work, and consists of eight Sections, as follows:—I., Additional Propositions; II., Centres of Similitude; III., Theory of Harmonic Section; IV., Theory of Inversion; V., Coaxal Circles; VI., Theory of Anharmonic Section; VII., Theory of Poles and Polars, and Reciprocation; VIII., Miscellaneous Exercises. Some of the Propositions in these Sections have first appeared in Papers published by myself; but the greater number have been selected from the writings of Chasles, Salmon, and Townsend. For the proofs given by those authors, in some instances others have been substituted, but in no case except where by doing so they could be made more simple and elementary.

I have to return my best thanks to Mr. Williamson, F.R.S., for many suggestions, as well as for his kindness in reading the proof sheets; and also to the Committee of the "Dublin University Press Series," for adopting my book as one of their publications.

JOHN CASEY.

2, Iona Terrace,
Dublin, *Feb.* 8, 1881.

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In these Exercises will be found many remarkable Propositions, some of which are of historical interest, such as Miquel's Theorem, Malfatti's Problem, Bellavitis's Theorem, and some others.

ERRATA.

Page 7, line 32, *insert* "is given."

,, 9, ,, 5, ,, Prop. *before* 12.

,, 9, ,, 15, *for* P *read* F.

,, 10, ,, 4, *insert* Prop. *before* 13.

,, 12, ,, 29, *for* $\perp$s *read* $\perp$.

,, 56, *Cor.* 2, *for* sum of squares of all the diagonals *read* the sum of the squares of all the lines of connection of its angular points.

,, 61, line 37, *for* $\angle$s *read* $\triangle$s.

,, 64, ,, 1, ,, circumscribed *read* escribed.

,, 67, ,, 26, ,, AC *read* A'C.

,, 73, ,, 4, ,, PO ,, PC.

,, 74, ,, 10, ,, $\frac{p'}{p''}$,, $\frac{p'}{p'''}$.

,, 78, ,, 21, ,, EC ,, E.

,, 80, ,, 21, *after* "construct" *insert* "it."

,, 90, ,, 32, *for* ABCD *read* A, B, C, D.

,, 99, ,, 1, *for* 7 *read* 6.

,, 107, last line, *insert* $\therefore$ *before* ON.

,, 108, line 23, *for* O *read* C.

,, 109, ,, 2, *for* AO *read* A, O.

,, 115, ,, 33, *for* $\odot$ *read* $\odot$s.

IN DIAGRAMS.

Page 117, join the points BE'.

,, 126, ,, CF, CE, AD, AE.

,, 128, ,, AP, A'P.

,, 131, second diagram: the point C should be joined to R instead of to M.

,, 140, join the points OC.

,, 146, ,, CC''.

The author will feel much obliged to readers who will communicate to him any mistakes they may find in this book, or who will give suggestions for its improvement.

BOOK FIRST.

SECTION I.

Additional Propositions.

In the following pages the Propositions of the text of Euclid will be referred to by Roman numerals enclosed in brackets, and those of the work itself by the Arabic. The number of the book will be given only when different from that under which the reference occurs.

For the purpose of saving space, the following contractions will be employed :—

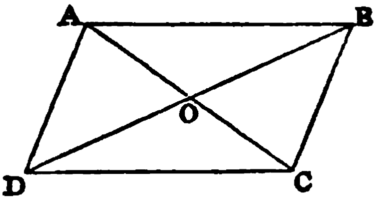
Circle will be denoted by		$\odot$
Triangle	,,	$\triangle$
Parallelogram	,,	▭
Parallel	,,	$\parallel$
Angle	,,	$\angle$
Perpendicular	,,	$\perp$

In addition to the foregoing, we shall use the usual symbols of Algebra, and other contractions whose meanings will be so obvious as not to require explanation.

Prop. 1.—*The diagonals of a ▭ bisect each other.*

Let ABCD be the ▭, its diagonals AC, BD bisect each other.

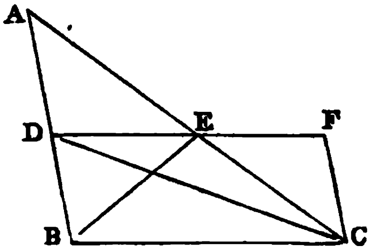
Dem.—Because AC meets the ||s AB, CD, the ∠BAO = ∠ DCO. In like manner, the ∠ABO = ∠CDO (xxix.), and the side AB = side CD (xxxiv.); ∴ AO = OC; BO = OD (xxvi.)



Cor. 1.—If the diagonals of a quadrilateral bisect each other it is a ▭.

Cor. 2.—If the diagonals of a quad[l]. divide it into four equal triangles, it is a ▭.

Prop. 2.—*The line* DE *drawn through the middle point* D *of the side* AB *of a* △, || *to a second side* BC, *bisects the third side* AC.

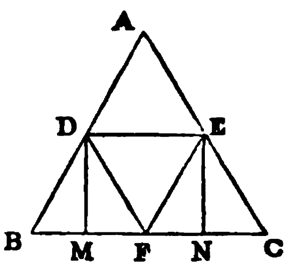


Dem.—Through C draw CF || to AB, meeting DE produced in F. Since BCFD is a ▭, CF = BD (xxxiv.); but BD = AD (hyp.); ∴ CF = AD. Again, the ∠ FCE = ∠ DAE, and ∠ EFC = ∠ ADE (xxix.); ∴ AE = EC (xxvi.). Hence AC is bisected.

Cor.—DE = $\frac{1}{2}$ BC. For DE = EF = $\frac{1}{2}$ DF.

Prop. 3.—*The line* DE *which joins the middle points* D *and* E *of the sides* AB, AC *of a* △ *is* || *to the base* BC.

Dem.—Join BE, CD, then △ BDE = △ ADE (xxxviii.), and △ CDE = △ ADE; therefore the △ BDE = △ CDE, and the line DE is || to BC (xxxix.)

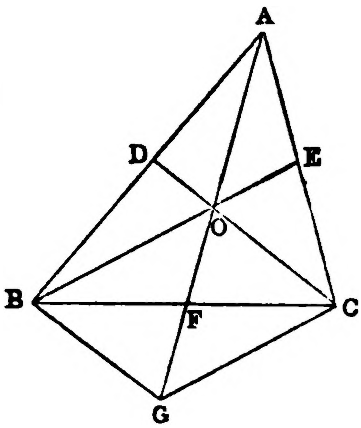


Cor. 1.—If D, E, F be the middle points of the sides AB, AC, BC of a △, the four △s into which the lines DE, EF, FD divide the △ ABC are all equal. This follows from (xxxiv.), because the figures ADFE, CEDF, BFED, are ▭s.

Cor. 2.—If through the points D, E, any two ||s be drawn meeting the base BC in two points M, N, the ▭ DENM is = ½ △ ABC. For DENM = ▭ DEFB (xxxv.).

Def.—*When three lines pass through the same point they are said to be concurrent.*

Prop. 4.—*The bisectors of the three sides of a △ are concurrent.*

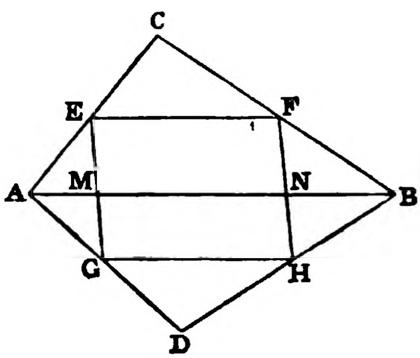


Let BE, CD, the bisectors of AC, AB, intersect in O, the Prop. will be proved by showing that AO produced bisects BC. Through B draw BG || to CD, meeting AO produced in G; join CG. Then, because DO bisects AB, and is || to BG, it bisects AG (2) in O. Again, because OE bisects the sides AG, AC, of the △ AGC, it is || to GC (3). Hence the figure OBGC is a ▭, and the diagonals bisect each other (1); ∴ BC is bisected in F.

Cor.—The bisectors of the sides of a △ divide each other in the ratio of 2 : 1.

Because AO = OG and OG = 2OF, AO = 2OF.

Prop. 5.—*The middle points* E, F, G, H *of the sides* AC, BC, AD, BD *of two △s* ABC, ABD, *on the same base* AB, *are the angular points of a ▭, whose area is equal to half sum or half difference of the areas of the △s, according as they are on opposite sides, or on the same side of the common base.*



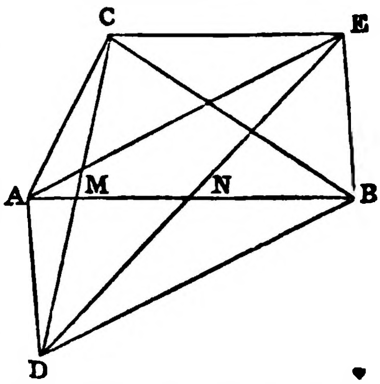
Dem. 1.—Let the △s be on opposite sides. The

figure EFGH is evidently a ▭, since the opposite sides EF, GH are each || to AB (3), and $= \frac{1}{2}$ AB (Prop. 2, *Cor.*). Again, let the lines EG, FH meet AB in the points M, N; then ▭ EFNM $= \frac{1}{2}$ △ ABC (Prop. 3, *Cor.* 2), and ▭ GHNM $= \frac{1}{2}$ △ ABD. Hence ▭ EFHG $= \frac{1}{2}$ (△ ABC + △ ABD).

Dem. 2.—When ABC, ABD are on the same side of AB, we have evidently ▭ EFGH = EFNM − GHNM $= \frac{1}{2}$(△ ABC − △ ABD).

Observation.—The second case of this proposition may be inferred from the first if we make the convention of regarding the sign of the area of the △ ABD to change from positive to negative, when the △ goes to the other side of the base. This affords a simple instance of a convention universally adopted by modern geometers, namely—when a geometrical magnitude of any kind, which varies continuously according to any law, passes through a zero value to give it the algebraic signs, plus and minus, on different sides of the zero; in other words, to suppose it to change sign in passing through zero, unless zero is a maximum or minimum.

Prop. 6.—*If two equal △s* ABC, ABD *be on the same base* AB, *but on opposite sides, the line joining the vertices* C, D *is bisected by* AB.

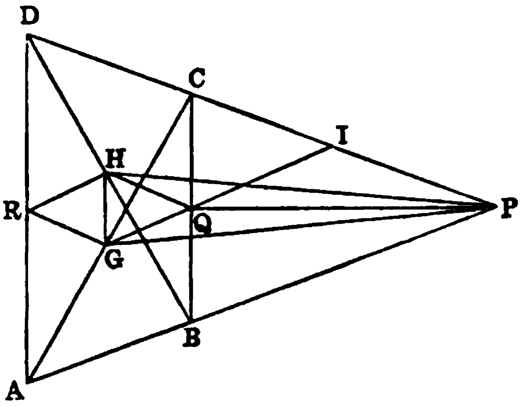


Dem.—Through A and B draw AE, BE || respectively to BD, AD; join EC. Now, since AEBD is a ▭, the △ AEB = △ADB (xxxiv.); but ADB = ACB (hyp.); ∴ △ AEB = △ACB; ∴ CE is || to AB (xxxix.) Let CD, ED meet AB in the points M, N, respectively. Now, since AEDB is a ▭, ED is bisected in N (1); and since NM is || to EC, CD is bisected in M (2).

Cor.—If the line joining the vertices of two △s on the same base, but on opposite sides, be bisected by the base, the △s are equal.

Prop. 7.—*If the opposite sides* AB, CD *of a quad'.*

meet in P, *and if* G, H *be the middle points of the diagonals* AC, BD, *the* △ PGH = ¼ *quad*¹. ABCD.



Dem.—Bisect the sides BC, AD in Q and R; join QH, QG, QP, RH, RG. Now, since QG is ∥ to AB (3), if produced it will bisect PC; then, since CP, joining the vertices of the △s CGQ, PGQ on the same base GQ, but on opposite sides, is bisected by GQ produced, the △ PGQ = △ CGQ (Prop. 6, *Cor.*) = ¼ △ ABC.

In like manner △ PHQ = ¼ △ BCD. Again, the ▭ GHQR = ½ (△ABD − △ ABC) (5); ∴ △ QGH = ¼ △ ABD − ¼ △ ABC: hence, △ PGH = ¼ (△ ABC + △ BCD + △ ABD − △ ABC) = ¼ quad¹. ABCD.

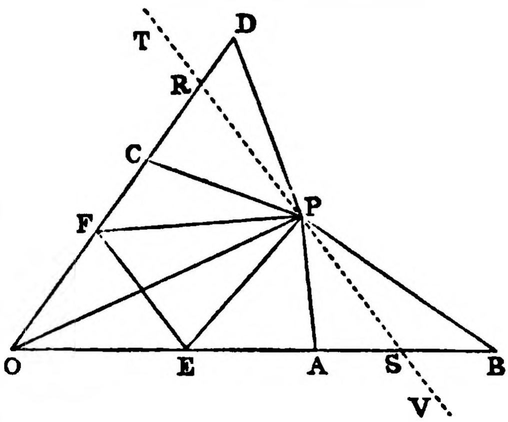
Cor.—The middle points of the three diagonals of a complete quad¹. are collinear (*i.e.* in the same right line). For, let AD and BC meet in S, then SP will be the third diagonal; join S and P to the middle points G, H of the diagonals AC, BD; then the △s SGH, PGH, being each = ¼ quad¹. ABCD, are = to one another; ∴ GH bisects SP (6).

Def.—*If a variable point moves according to any law, the path which it describes is termed its locus.*

Thus, if a point P moves so as to be always at the same distance from a fixed point O, the locus of P is a ⊙, whose centre is O and radius = OP. Or, again, if A and B be two fixed points, and if a variable point P moves so that the area of the △ ABP retains the same value during the motion, the locus of P will be a right line ∥ to AB.

Prop. 8.—*If* AB, CD *be two lines given in position and magnitude, and if a point* P *moves so that the sum of the areas of the* △*s* ABP, CDP *is given, the locus of* P *is a right line.*

Dem.—Let AB, CD intersect in O; then cut off OE = AB, and OF = CD; join OP, EP, EF, FP; then △ APB = △ OPE, and △ CPD = △ OFF; hence the sum of the areas of the △s OEP, OFP is given; ∴ the area of the quad[l]. OEPF is given; but the △ OEF is evidently given; ∴ the area of the △ EFP is given, and the base EF is given; ∴ the locus of P is a right line ∥ to EF.



Let the locus in this question be the punctured line in the diagram. It is evident when the point P coincides with R, the area of the △ CDP vanishes; and when the point P passes to the other side of CD, such as to the point T, the area of the △ CDP must be regarded as negative. Similar remarks hold for the △ APB and the line AB. This is an instance of the principle (see 5, note) that the area of a △ passes from positive to negative as compared with any given △ in its own plane, when (in the course of any continuous change) its *vertex crosses its base*.

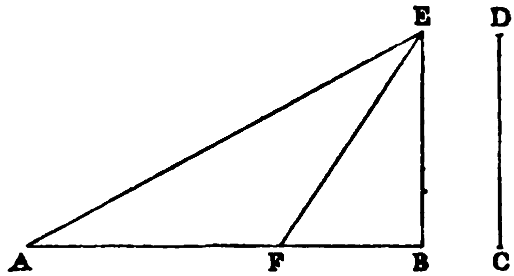
Cor. 1.—If m and n be any two multiples, and if we make OE = mAB and OF = nCD, we shall in a similar way have the locus of the point P when m times △ ABP + n times △ CDP is given; viz., it will be a right line ∥ to EF.

Cor. 2.—If the line CD be produced through O, and if we take in the line produced, OF′ = nCD, we shall get the locus of P when m times △ ABP − n times △ CDP is given.

Cor. 3.—If three lines, or in general any number of lines, be given in magnitude and position, and if m, n, p, q, &c., be any system of multiples, all positive, or some positive and some negative, and if the area of m times △ ABP + n times △ CDP + p times △ GHP + &c., be given, the locus of P is a right line.

Cor. 4.—If ABCD be a quad[l]., and if P be a point, so that the sum of the areas of the $\triangle$s ABP, CDP is half the area of the quad[l]., the locus of P is a right line passing through the middle points of the three diagonals of the quad[l].

Prop. 9.—*To divide a given line* AB *into two parts, the difference of whose squares shall be = to the square of a given line* CD.



Con.—Draw BE at right angles to AB, and make it = CD; join AE, and make the $\angle$ AEF = $\angle$ EAB, then F is the point required.

Dem.—Because the $\angle$ AEF = $\angle$ EAF, the side AF = FE; $\therefore AF^2 = FE^2 = FB^2 + BE^2$; $\therefore AF^2 - FB^2 = BE^2$, but $BE^2 = CD^2$; $\therefore AF^2 - FB^2 = CD^2$.

If CD be greater than AB, BE will be greater than AB, and the $\angle$ EAB will be greater than the $\angle$ AEB; hence the line EF, which makes with AE the $\angle$ AEF = $\angle$ EAB, will fall at the other side of EB, and the point F will be in the line AB produced. The point F is in this case a point of external division.

Prop. 10.—*Given the base of a* $\triangle$ *in magnitude and position, and given also the difference of the squares of its sides, to find the locus of its vertex.*

Let ABC be the $\triangle$ whose base AB is given; let fall the $\perp$ CP on AB; then

$$AC^2 = AP^2 + CP^2; \qquad \text{(xlvii.)}$$
$$BC^2 = BP^2 + CP^2;$$
$$\therefore AC^2 - BC^2 = AP^2 - BP^2;$$

but $AC^2 - BC^2$ is given; $\therefore AP^2 - BP^2$. Hence AB is divided in P into two parts, the difference of whose squares is given; $\therefore$ P is a given point (9), and the line CP is given in position; and since the point C must be always on the line CP, the locus of C is a right line $\perp$ to the base.

Cor.—The three $\perp$s of a $\triangle$ are concurrent. Let

the $\perp$s from A and B on the opposite sides be AD and BE, and let O be the point of intersection of these $\perp$s.

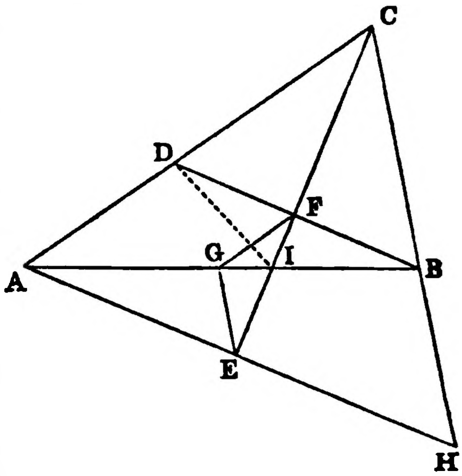
Now, $\quad AC^2 - AB^2 = OC^2 - OB^2$; (10)

and $\quad AB^2 - BC^2 = OA^2 - OC^2$;

$\therefore AC^2 - BC^2 = OA^2 - OB^2$.

Hence the line CO produced will be $\perp$ to AB.

Prop. 11.—*If* $\perp$*s* AE, BF *be drawn from the extremities* A, B *of the base of a* $\triangle$ *on the internal bisector of the vertical* $\angle$, *the line joining the middle point* G *of the base to the foot of either* $\perp$ *is = to half the difference of the sides* AC, BC.



Dem.—Produce BF to D; then in the $\triangle$s BCF, DCF there are evidently two $\angle$s, and a side of one = respectively to two $\angle$s, and a side of the other; $\therefore$ CD = CB and FD = FB; hence AD is the difference of the sides AC, BC; and, since F and G are the middle points of the sides BD, BA, FG = $\frac{1}{2}$ AD = $\frac{1}{2}$ (AC − BC). In like manner EG = $\frac{1}{2}$ (AC − BC).

Cor. 1.—In like manner it may be proved that lines drawn from the middle point of the base to the feet of $\perp$s from the extremities of the base on the bisector of the external vertical angle is = half sum of AC and BC.

Cor. 2.—The $\angle$ ABD is = $\frac{1}{2}$ difference of the base $\angle$s.

Cor. 3.—The $\angle$ CBD is = half sum of the base $\angle$s.

Cor. 4.—The $\angle$ between CF and the $\perp$ from C on AB = $\frac{1}{2}$ difference of base $\angle$s.

Cor. 5.—The $\angle$ AID = difference of base $\angle$s.

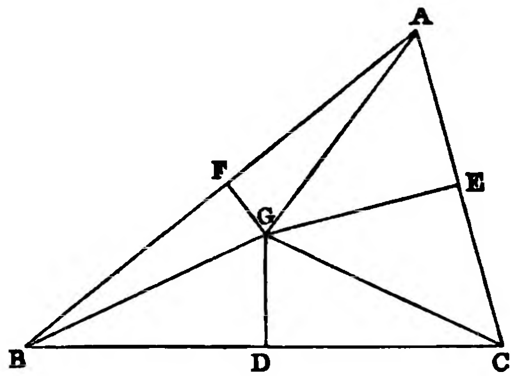
Cor. 6.—Given the base and the difference of the sides of a $\triangle$, the locus of the feet of the $\perp$s from the extremities of base on the bisector of the internal vertical $\angle$ is a circle, whose centre is the middle point of the base and radius = half difference of the sides.

Cor. 7.—Given the base of a $\triangle$ and the sum of the

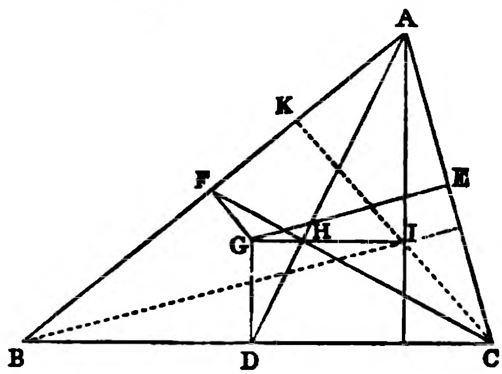
sides, the locus of the feet of the ⊥s from the extremities of the base on the bisector of the external vertical ∠ is a ⊙, whose centre is the middle point of the base and radius = half sum of the sides.

12. *The three ⊥s to the sides of a △ at their middle points are concurrent.*

Dem.—Let the middle points be D, E, F. Draw FG, EG ⊥ to AB, AC, and let these ⊥s meet in G; join GD; the prop. will be proved by showing that GD is ⊥ to BC; join AG, BG, CG. Now, in the △s APG and BFG, since AF = FB, and FG common, and the ∠ AFG = ∠ BFG, AG is = GB (iv.). In like manner AG = GC; hence BG = GC. And since the △s BDG, CDG have the side BD = DC and DG common, and the base BG = GC, the ∠ BDG = ∠ CDG (viii.); ∴ GD is ⊥ to BC.



Cor. 1.—If the bisectors of the sides of the △ meet in H, and GH be joined and produced to meet any of the three ⊥s from the ∠s on the opposite sides; for instance, the ⊥ from A to BC, in the point I, suppose; then GH = ½ HI. For DH = ½ HA (*Cor.*, Prop. 4).



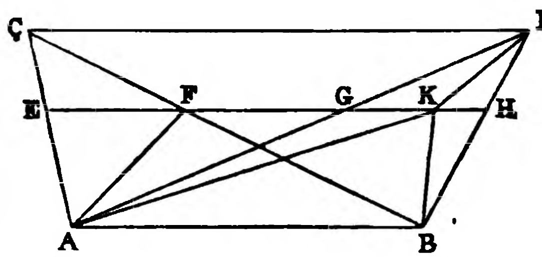
Cor. 2.—Hence the ⊥s of the △ pass through the point I. This is another proof that the ⊥s of a △ are concurrent.

Cor. 3.—The lines GD, GE, GF are respectively = ½ IA, ½ IB, ½ IC.

Cor. 4.—The point of concurrence of ⊥s from the

∠s on the opposite sides, the point of concurrence of bisectors of sides, and the point of concurrence of ⊥s at middle points of sides of a △ are collinear.

13. *If two △s* ABC, ABD, *be on the same base* AB *and between the same* || *s, and if a* || *to* AB *intersect the lines* AC, BC, *in* E *and* F, *and the lines* AD, BD, *in* G *and* H, EF *is* = GH.

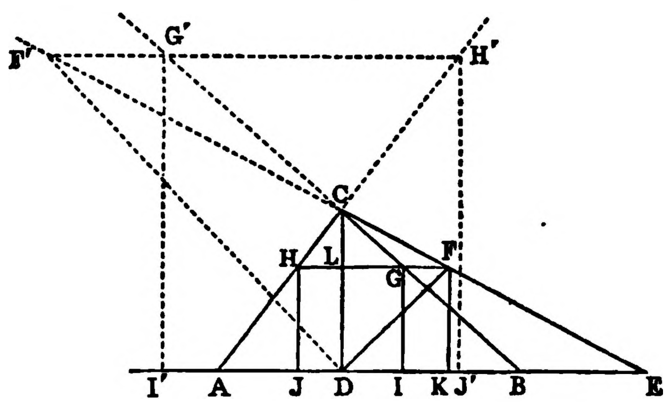


Dem.—If not, let GH be greater than EF, and cut off GK = EF. Join AK, KB, KD, AF; then (xxxviii.) △ AGK = △ AEF, and △ DGK = △ CEF, and (xxxvii.) △ ABK = △ ABF; ∴ quad^1. ABKD = △ ABC; but △ ABC = △ ABD; ∴ quad^1. ABKD = △ ABC, which is impossible. Hence EF = GH.

Cor. 1.—If instead of two △s on the same base and between the same || s, we have two △s on = bases and between the same || s, the intercepts made by the sides of the △s on a || to the line joining the vertices are equal.

Cor. 2.—The line drawn from the vertex of a △ to the middle point of the base bisects any line parallel to the base, and terminated by the sides of the △.

Prop. 14.—*To inscribe a square in a* △.



Con.—Let ABC be the △: let fall the ⊥ CD; cut off BE = AD; join EC; bisect the ∠ EDC by the line DF,

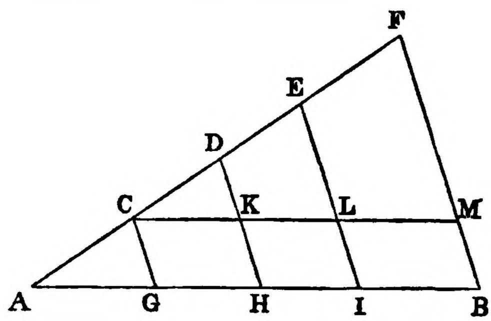
meeting EC in F; through F draw a || to AB, cutting the sides BC, AC in the points G, H; from G H let fall the ⊥s GI, HJ: the figure GIHJ is a square.

Dem.—Since the ∠ EDC is bisected by DF, and the ∠s K and L right angles, and DF common, FK = FL (xxvi.); but FL = GH (Prop. 13, *Cor.* 1), and FK = GI (xxxiv.); ∴ GI = GH, and the figure IGHJ is a square, and it is inscribed in the △.

Cor.—If we bisect the ∠ ADC by the line DF, meeting EC produced in F′, and through F′ draw a line || to AB meeting BC, and AC produced in G′, H′, and from G′, H′ let fall ⊥s G′ I′, H′ J′ on AB, we shall have an escribed square.

Prop. 15.—*To divide a given line* AB *into any number of equal parts.*

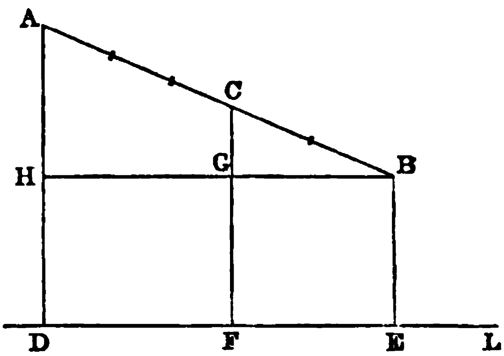
Con.—Draw through A any line AF, making an ∠ with AB; in AF take any point C, and cut off CD, DE, EF, &c., each = AC, until we have as many = parts as the number into which we want to divide AC; say, for instance, four = parts. Join BF; and draw CG, DH, EI, each || to BF; then AB is divided into four = parts.



Dem.—Since ADH is a △, and AD is bisected in C, and CG is || to DH; then (2) AH is bisected in G; ∴ AG = GH. Again, through C draw a line || to AB, cutting DH and EI in K and L; then, since CD = DE, we have (2) CK = KL; but CK = GH, and KL = HI; ∴ GH = HI. In like manner, HI = IB. Hence the parts into which AB is divided are all equal.

This Proposition may be enunciated as a theorem as follows:—If one side of a △ be divided into any number of equal parts, and through the points of division lines be drawn || to the base, these ||s will divide the second side into the same number of = parts.

Prop. 16.—*If a line* AB *be divided into* $(m+n) =$ *parts, and suppose* AC *contains* m *of these parts,* CB *contains* n *of them. Then, if from the points* A, C, B $\perp$*s be let fall on any line, such as* AD, CF, BE, *then* $m\,\text{BE} + n\,\text{AD} = (m+n)\,\text{CF}$.



Dem. — Draw BH $\parallel$ to ED, and through the points of division of AB imagine lines drawn $\parallel$ to BH; these lines will divide AH into $m+n$ = parts, and CG into n = parts; $\therefore$ n times AH $= (m+n)$ times CG; and since DH and BE are each = GF, we have n times HD + m times BE $= (m+n)$ times GF. Hence, by addition, n times AD + m times BE $= (m+n)$ times CF.

Def.—*The centre of mean position of any number of points* A, B, C, D, *&c., is a point which may be found as follows:—Bisect the line joining any two points* AB *in* G, *join* G *to a third point* C, *and divide* GC *in* H, *so that* $\text{GH} = \frac{1}{3}\,\text{GC}$; *join* H *to a fourth point* D, *and divide* HD *in* K, *so that* $\text{HK} = \frac{1}{4}\,\text{HD}$, *and so on: the last point found will be the centre of mean position of the system of points.*

Prop 17.—*If there be any system of points* A, B, C, D, *whose number is* n, *and if* $\perp$*s be let fall from these points on any line* L, *the sum of the* $\perp$*s from all the points on* L *is* $= n$ *times the* $\perp$*s from the centre of mean position.*

Dem.—Let the $\perp$s be denoted by AL, BL, CL., &c. Then, since AB is bisected in G, we have (16)

$$\text{AL} + \text{BL} = 2\text{GL};$$

and since GC is divided into $(1+2) =$ parts in H, so that HG contains one part and HC two parts; then $2\text{GL} + \text{CL} = 3\text{HL}$;

$$\therefore\ \text{AL} + \text{BL} + \text{CL} = 3\text{HL}, \text{ \&c., \&c.}$$

Hence the Proposition is proved.

Cor.—If from any number of points $\perp$s be let fall on any line passing through their mean centre, the sum of the $\perp$s is zero. Hence some of the $\perp$s must be negative, and we have the sum of the $\perp$s on the positive side = the sum of those on the negative side.

Prop. 18.—*We may extend the foregoing Definition as follows:—Let there be any system of points* A, B, C, D, *&c., and a corresponding system of multiples a, b, c, d, &c., connected with them, then divide the line joining the points* AB *into $(a + b) =$ parts, and let* AG *contain b of these parts, and* GB *contain a parts. Again, join* G *to a third point* C, *and divide* GC *into $(a + b + c) =$ parts, and let* GH *contain c of these parts, and* HC *the remaining parts, and so on; then the point last found will be the mean centre for the system of multiples a, b, c, d, &c.*

From this Definition we may prove exactly the same as in Prop. 17, that if AL, BL, CL, &c., be the $\perp$s from the points A, B, C, &c., on any line L, then

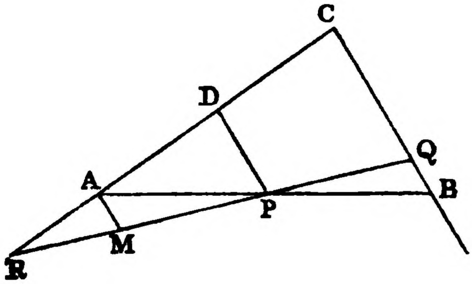
$$a \,.\, \mathrm{AL} + b \,.\, \mathrm{BL} + c \,.\, \mathrm{CL} + d \,.\, \mathrm{DL} + \&c.$$

$= (a + b + c + d + \&c.)$ times the $\perp$ from the centre of mean position on the line L.

Def.—*If a geometrical magnitude varies its position continuously according to any law, and if it retains the same value throughout, it is said to be a constant; but if it goes on increasing for some time, and then begins to decrease, it is said to be a maximum at the end of the increase: again, if it decreases for some time, and then begins to increase, it is a minimum when it commences to increase.*

From these Definitions it will be seen that a maximum value is greater than the ones which immediately precede and follow it; and that a minimum is less than the value of that which immediately precedes, and less than that which immediately follows. We give here a few simple but important Propositions bearing on this part of Geometry.

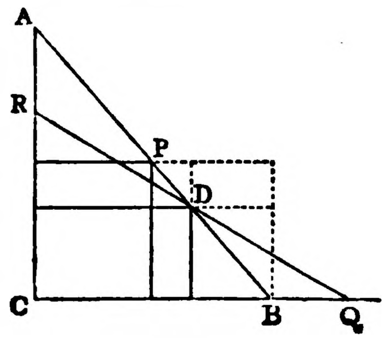
Prop. 19—*Through a given point* P *to draw a line which shall form, with two given lines* CA, CB, *a* △ *of minimum area.*



Con.—Through P draw PD ‖ to CB; cut off AD = CD; join AP, and produce to B. Then AB is the line required.

Dem.—Let RQ be any other line through P; draw AM ‖ to CB. Now, because AD = DC, we have AP = PB; and the △s APM and QPB have the ∠s APM, AMP respectively = to the ∠s BPQ, BQP, and the sides AP and PB = to one another; ∴ the △s are equal; hence the △ APR is greater than the △ BPQ: to each add the quad[l]. CAPQ, and we get the △ CQR greater than the △ ABC.

Cor. 1.—The line through the point P which cuts off the minimum △ is bisected in that point.



Cor. 2.—If through the middle point P, and through any other point D of the side AB of the △ ABC we draw lines ‖ to the remaining sides, so as to form two inscribed ▭s CP, CD, then CP is greater than CD.

Dem. — Through D draw QR, so as to be bisected in D; then the △ ABC is greater than the △ CQR; but the ▭s are halves of the △s; hence CP is greater than CD.

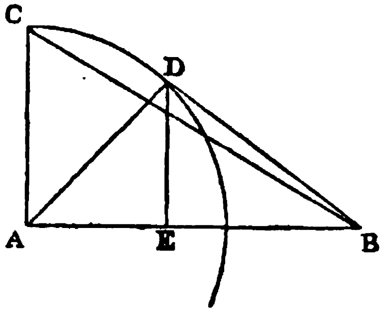
A very simple proof of this *Cor.* can also be given by means of (xliii.)

Prop. 20.—*When two sides of a* △ *are given in magnitude, the area is a maximum when they contain a right angle.*

Dem.—Let BAC be a △ having the ∠ A right, with A as centre and AC as radius; describe a ⊙; take any other point D in the circumference; it is

evident the Prop. will be proved by showing that the △ BAC is greater than BAD.

Let fall the ⊥ DE; then (xix.) AD is greater than DE; ∴ AC is greater than DE; and since the base AB is common, the △ ABC is greater than the △ ABD.

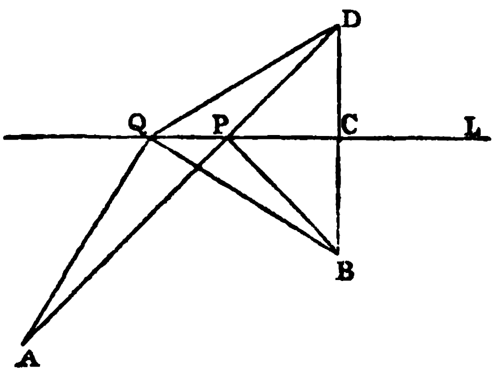


Cor.—If the diagonals of a quad[l]. be given in magnitude, the area is a maximum when they are at right angles to each other.

Prop. 21.—*Given two points,* A, B: *it is required to find a point* P *in a given line* L, *so that* AP + PB *may be a minimum.*

Con.—From B let fall the ⊥ BC on L; produce BC to D, and make CD = CB; join AD, cutting L in P; then P is the point required.

Dem.—Join PB, and take any other point Q in L; join AQ, QB, QD. Now, since BC = CD and CP common, and the ∠s at C right ∠s, we have BP = PD. In like manner BQ = QD; to these equals add respectively AP and AQ, and we have AD = AP + PB, and AQ + QD = AQ + QB; but AQ + QD is greater than AD; ∴ AQ + QB is greater than AP + PB.



Cor. 1.—The lines AP, PB, whose sum is a minimum, make equal angles with the line L.

Cor. 2.—The perimeter of a variable △, inscribed in a fixed △, is a minimum when the sides of the former make = ∠s with the sides of the latter. For, suppose one side of the inscribed △ to remain fixed while the two remaining sides vary, the sum of the varying sides will be a minimum when they make = ∠s with the side of the fixed △.

Cor. 3.—Of all polygons whose vertices lie on fixed lines, that of minimum perimeter is the one whose several $\angle$s are bisected externally by the lines on which they move.

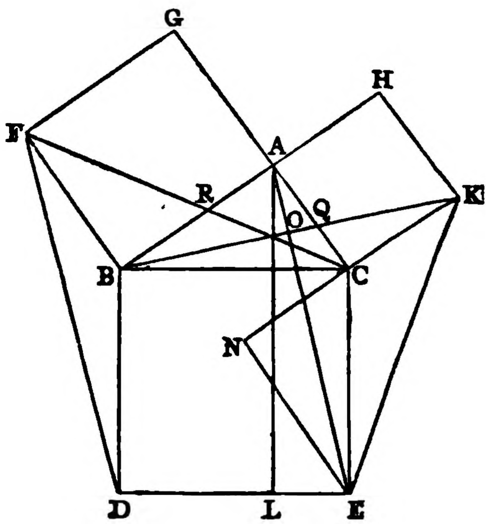
Prop. 22.—*Of all $\triangle$s having the same base and area, the perimeter of an isosceles $\triangle$ is a minimum.*

Dem.—Since the $\triangle$s are all = in area, the vertices must lie on a line $\parallel$ to the base, and the sides of an isosceles $\triangle$ will evidently make = $\angle$s with this parallel; hence their sum is a minimum.

Cor.—Of all polygons having the same number of sides and equal areas, the perimeter of an equilateral polygon is a minimum.

Prop. 23.—*A large number of deducibles may be given in connexion with Euclid,* fig., Prop. xlvii. *We insert a few here, confining ourselves to those that may be proved by the First Book.*

(1). The transverse lines AE, BK are $\perp$ to each other. For, in the $\triangle$s ACE, BCK, which are in every respect =, the $\angle$ EAC = $\angle$ BCK, and the $\angle$ AQO = $\angle$ KQC; hence the $\angle$ AOQ = $\angle$ KCQ, and $\therefore$ a right angle.



(2). $\triangle$ KCE = $\triangle$ DBF.

Dem.—Produce KC, and let fall the $\perp$ EN. Now, the $\angle$ ACN = BCE, each being right; $\therefore$ $\angle$ ACB = $\angle$ ECN, and $\angle$ BAC = $\angle$ ENC, each being right, and side BC = CE; hence (xxvi.) EN = AB and CN = AC; but AC = CK; $\therefore$ CN = CK, and $\triangle$ ENC = $\triangle$ ECK (xxxviii.); but $\triangle$ ENC = $\triangle$ ABC; hence $\triangle$ ECK = $\triangle$ ABC. In like manner, $\triangle$ DBF = $\triangle$ ABC; $\therefore$ $\triangle$ ECK = $\triangle$ DBF.

(3). $EK^2 + FD^2 = 5BC^2$.

Dem.—$EK^2 = EN^2 + NK^2$ (xlvii.);

but $\quad EN = AB$, and $NK = 2AC$;

$$\therefore EK^2 = AB^2 + 4AC^2.$$

In like manner

$$FD^2 = 4AB^2 + AC^2;$$

$$\therefore EK^2 + FD^2 = 5(AB^2 + AC^2) = 5BC^2.$$

(4). The intercepts AQ, AR are equal.

(5). The lines CE, BK, AL are concurrent.

SECTION II.

Exercises.

1. The line which bisects the vertical ∠ of an isosceles △ bisects the base perpendicularly.

2. The diagonals of a quad[l]. whose four sides are equal bisect each other perpendicularly.

3. The common chord of two ⊙s is bisected perpendicularly by the line joining their centres.

4. From a given point in one of the sides of a △ draw a right line bisecting the area of the △.

5. The sum of the ⊥s from any point in the base of an isosceles △ on the equal sides is = to the ⊥ from one of the base angles on the opposite side.

6. If the point be taken in the base produced, prove that the difference of the ⊥s on the equal sides is = to the ⊥ from one of the base angles on the opposite side; and show that, having regard to the convention respecting the signs plus and minus, this theorem is a case of the last.

7. If the base BC of a △ be produced to D, the ∠ between the bisectors of the ∠s ABC, ACD = half ∠ BAC.

8. The bisectors of the three internal angles of a △ are concurrent.

9. Any two external bisectors and the third internal bisector of the angles of a △ are concurrent.

10. The quad[ls]. formed either by the four external or the four internal bisectors of the angles of any quad[l]. have their opposite ∠s = two right ∠s.

11. Draw a right line ∥ to the base of a △, so that

(1). Sum of lower segments shall be = to a given line.

(2). Their difference shall be = to a given line.

(3). The ∥ shall be = sum of lower segments.

(4). The ∥ shall be = difference of lower segments.

12. If two lines be respectively ⊥ to two others, the ∠ between the former is = to the ∠ between the latter.

13. If two lines be respectively ∥ to two others, the ∠ between the former is = to the ∠ between the latter.

14. The △ formed by the three bisectors of the external angles of a △ is such that the lines joining its vertices to the ∠s of the original △ will be its ⊥s.

15. From two points on opposite sides of a given line it is required to draw two lines to a point in the line, so that their difference will be a maximum.

16. State the converse of Prop. xvii.

17. Give a direct proof of Prop. xix.

18. Given the lengths of the bisectors of the three sides of a △: construct it.

19. If from any points ⊥s be drawn to the three sides of a △, prove that the sum of the squares of three alternate segments of the sides = the sum of squares of the three remaining segments.

20. Prove the following theorem, and infer from it Prop. xlvii.: If CQ, CP be ▭s described on the sides CA, CB of a △, and if the sides ∥ to CA, CB be produced to meet in R, and RC joined, a ▭ described on AB with sides = and ∥ to RC shall be = to the sum of the ▭s CQ, CP.

21. If a square be inscribed in a △, the rectangle under its side and the sum of base and altitude = twice the area of the △.

22. If a square be escribed to a △, the rectangle under its side and the difference of the base and altitude = twice the area of the △.

23. Given the difference between the diagonal and side of a square: construct it.

24. The sum of the squares of lines joining any point in the plane of a rectangle to one pair of opposite angular points = sum of the squares of the lines drawn to the two remaining angular points.

25. If two lines be given in position, the locus of a point equidistant from them is a right line.

26. In the same case the locus of a point, the sum or the difference of whose distances from them is given, is a right line.

27. Given the sum of the perimeter and altitude of an equilateral $\triangle$: construct it.

28. Given the sum of the diagonal and two sides of a square: construct it.

29. From the extremities of the base $\angle$s of an isosceles $\triangle$ right lines are drawn $\perp$ to the sides, prove that the base $\angle$s of the $\triangle$ are each = half the $\angle$ between the $\perp$s.

30. The line joining the middle point of the hypotenuse of a right-angled triangle to the right angle is = half the hypotenuse.

31. The lines joining the feet of the $\perp$s of a $\triangle$ form an inscribed $\triangle$ whose perimeter is a minimum.

32. If from the extremities A, B of the base of a $\triangle$ ABC $\perp$s AD, BE be drawn to the opposite sides, prove that

$$AB^2 = AC \cdot AE + BC \cdot BD.$$

33. If A, B, C, D, &c., be any number (n) of points on a line, and O their centre of mean position; then, if P be any other point on the line,

$$AP + BP + CP + DP + \&c. = nOP.$$

34. If O, O′ be the centres of mean position for two systems of collinear points A, B, C, D, &c., A,′ B,′ C,′ D,′ &c., each system having the same number (n) of points, then

$$nOO' = AA' + BB' + CC' + DD' + \&c.$$

35. If G be the point of intersection of the bisectors of the $\angle$s A, B of a $\triangle$, right-angled at C, and GD a $\perp$ on AB, then, the rectangle AD . DB = area of the $\triangle$.

36. The sides AB, AC of a $\triangle$ are bisected in D, E; CD; BE intersect in F: prove $\triangle$ BFC = quad[l]. ADFE.

37. If lines be drawn from a fixed point to all the points of the circumference of a given $\odot$, the locus of all their points of bisection is a $\odot$.

38. Show by drawing $\parallel$ lines how to construct a $\triangle$ = to any given rectilineal figure.

39. ABCD is a $\square$: show that if B be joined to the middle point of CD, and D to the middle point of AB, the joining lines will trisect AC.

40. The equilateral $\triangle$ described on the hypotenuse of a right-angled $\triangle$ = sum of equilateral $\triangle$s described on the sides.

41. If squares be described on the sides of any $\triangle$, and the adjacent corners joined, the three $\triangle$s thus formed are equal.

42. If AB, CD be opposite sides of a ▭, P any point in its plane; then $\triangle$ PBC = sum or difference of the $\triangle$s CDP, ABP, according as P is outside or between the lines AC and BD.

43. If equilateral $\triangle$s be described on the sides of a right-angled $\triangle$ whose hypotenuse is given in magnitude and position, the locus of the middle points of the line joining their vertices is a $\odot$.

44. If CD be a $\perp$ on the base AB of a right-angled $\triangle$ABC, and if E, F be the centres of the $\odot$s described in the $\triangle$s ACD, BCD, and if EG, FH be lines through E and F $\parallel$ to CD, meeting AC, BC in G, H; then CG = CH.

45. If A, B, C, D, &c., be any system of collinear points, OP their mean centre for the system of multiples a, b, c, d, &c.; then if P be any other point in the line,

$$(a+b+c+d+\&c)\ \mathrm{OP} = a\,.\,\mathrm{AP} + b\,.\,\mathrm{BP} + c\,.\,\mathrm{CP} + d\,.\,\mathrm{DP} + \&c.$$

46. If O, O′ be the mean centres of the two systems of points A, B, C, D, &c., A′, B′, C′, D′, on the same line L for a common system of multiples a, b, c, d, &c.; then

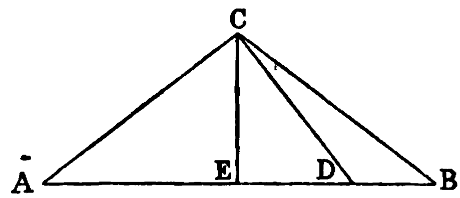
$$(a+b+c+d+\&c.)\,.\,\mathrm{OO} = a\,.\,\mathrm{AA'} + b\,.\,\mathrm{BB'} + c\,.\,\mathrm{CC'} + d\,.\,\mathrm{OO'}\ \&c.$$

BOOK SECOND.

SECTION I.

Additional Propositions.

Prop. 1.—*If* ABC *be an isosceles* △, *whose equal sides are* AC, BC; *if* CD *be a line drawn from* C *to any point* D *in the base* AB; *then* $AD \cdot DB = BC^2 - CD^2$.



Dem.—Let fall the ⊥ CE; then AB is bisected in E and divided unequally in E;

$$\therefore AD \cdot DB + ED^2 = EB^2; \qquad \text{(v.)}$$

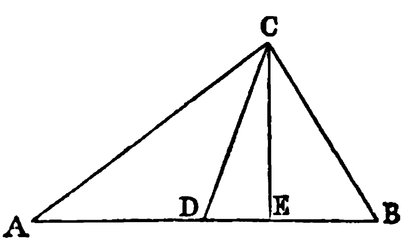
adding to each side EC^2;

$$\therefore AD \cdot DB + CD^2 = BC^2; \qquad \text{(I. xlvii.)}$$

$$\therefore AD \cdot DB = BC^2 - CD^2.$$

Cor.—If the point be in the base produced, we shall have $AD \cdot BD = CD^2 - CB^2$. If we consider that DB changes its sign when D passes through B, we see that this case is included in the last.

Prop. 2.—*If* ABC *be any* △, D *the middle point of* AB, $AB^2 + BC^2 = 2AD^2 + 2DC^2$.



Dem.—Let fall the $\perp$ EC.

$$AB^2 = AD^2 + DC^2 + 2AD \cdot DE; \qquad \text{(xii.)}$$
$$BC^2 = BD^2 + DC - 2DB \cdot DE. \qquad \text{(xiii.)}$$

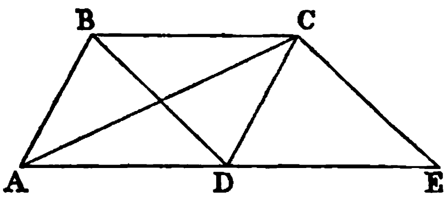
Hence, by addition, since

$$AD = DB,\ AC^2 + BC^2 = 2AD^2 + 2DC^2.$$

This is a simple case of a very general Prop., which we shall prove, on the properties of the centre of mean position for any system of points and any system of multiples. The Props. ix. and x. of the Second Book are particular cases of this Prop., viz., when the point C is in the line AB or the line AB produced.

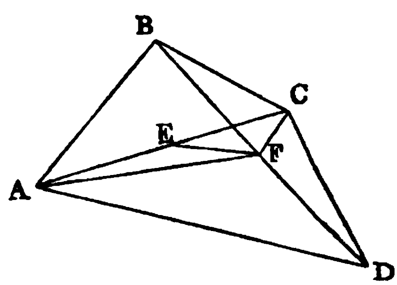
Cor.—If the base of a $\triangle$ be given, both in magnitude and position, and the sum of the squares of the sides in magnitude, the locus of the vertex is a $\odot$.

Prop. 3.—*The sum of the squares of the diagonals of a ▭ is equal to the sum of the squares of its four sides.*



Dem.—Let ABCD be the ▭. Draw CE ∥ to BD; produce AD to meet CE. Now, AD = BC (xxxiv.), and DE = BC; $\therefore$ AD = DE; hence (2) $AC^2 + CE^2 = 2AD^2 + 2DC^2$; but $CE^2 = BD^2$; $\therefore AC^2 + BD^2 = 2AD^2 + 2DC^2$ = sum of squares of the four sides of the ▭.

Prop. 4.—*The sum of the squares of the four sides of a quad^l. = sum of squares of its diagonals + four times the square of the line joining the middle points of the diagonals.*



Dem.—Let ABCD be the quad^l., E, F the middle points of the diagonals.

Now, $\quad AB^2 + AD^2 = 2AF^2 + 2FB^2, \qquad (2)$

and $\quad BC^2 + CD^2 = 2CF^2 + 2FB^2; \qquad (2)$

$$\therefore AB^2 + BC^2 + CD^2 + DA^2 = 2(AF^2 + CF^2) + 4FB^2$$
$$= 4AE^2 + 4EF^2 + 4FB^2 = AC^2 + BD^2 + 4EF^2.$$

Prop. 5.—*Three times the sum of the squares of the sides of a △ is equal to four times the sum of the squares of the lines bisecting the sides of the △.*

Dem.—Let D, E, F be the middle points of the sides.

Then $\quad AB^2 + AC^2 = 2BD^2 + 2DA^2;$ (2)

$\therefore\ 2AB^2 + 2AC^2 = 4BD^2 + 4DA^2;$

that is $\quad 2AB^2 + 2AC^2 = BC^2 + 4DA^2.$

Similarly $\quad 2BC^2 + 2BA^2 = CA^2 + 4EB^2;$

and $\quad 2CA^2 + 2AB^2 = AB^2 + 4FC^2.$

Hence $\quad 3(AB^2 + BC^2 + CA^2) = 4(AD^2 + BE^2 + CF^2).$

Cor.—If G be the point of intersection of the bisectors of the sides, $3AG = 2AD$. Hence $9AG^2 = 4AD^2$;

$$\therefore\ 3(AB^2 + BC^2 + CA^2) = 9(AG^2 + BG^2 + CG^2);$$
$$\therefore\ (AB^2 + BC^2 + CA^2) = 3(AG^2 + BG^2 + CG^2).$$

Prop. 6.—*The rectangle contained by the sum and difference of two sides of a △ is equal to twice the rectangle contained by the base, and the intercept between the middle point of the base and the foot of the ⊥ from the vertical ∠ on the base.*

Let CD be the ⊥ and E the middle point of AB.

Then $\quad AC^2 = AD^2 + DC^2,$

and $\quad BC^2 = BD^2 + DC^2;$

$\therefore\ AC^2 - BC^2 = AD^2 - DB^2;$

or $\quad (AC + BC)(AC - BC) = (AD + DB)(AD - DB).$

Now, $\quad AD + DB = AB$, and $AD - DB = 2ED$;

$\therefore\ (AC + BC)(AC - BC = 2AB \,.\, ED.$

Prop. 7.—*If* A, B, C, D *be four points taken in order on a right line, then* $AB \,.\, CD + BC \,.\, AD = AC \,.\, BD$.

Dem.—Let $AB = a$, $BC = b$, $CD = c$; then $AB \,.\, CD + BC \,.\, AD = ac + b(a + b + c) = (a + b)(b + c) = AC \,.\, BD.$

This theorem, which is due to Euler, is one of the most important in Elementary Geometry. It may be written in a more symmetrical form by making use

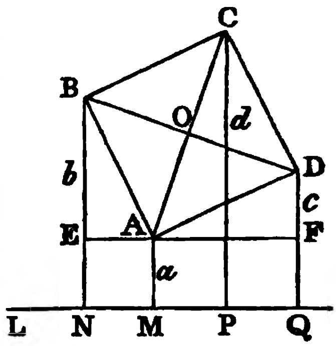
of the convention regarding + and −: thus, since $+AC = -CA$, we get

$$AB \,.\, CD + BC \,.\, AD = -CA \,.\, BD,$$

or

$$AB \,.\, CD + BC \,.\, AD + CA \,.\, BD = 0.$$

Prop. 8.—*If ⊥s be drawn from the angular points of a square to any line, the sum of the squares of the ⊥s from one pair of opposite ∠s exceeds twice the rectangle of the ⊥s from the other pair of opposite ∠s by the area of the square.*



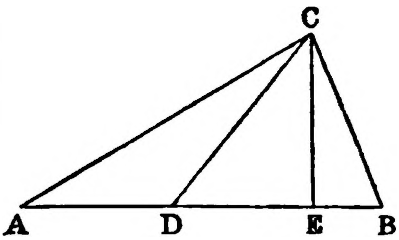
Dem. — Let ABCD be the square, L the line; let fall the ⊥s AM, BN, CP, DQ, on L: through A draw EF ∥ to L. Now, since the ∠ BAD is right, the ∠s BAE, DAF = one right ∠, and ∴ = the ∠s BAE, ABE; ∴ ∠ ABE = ∠DAF, and ∠ E = ∠ F, and AB = AD, ∴ AE = DF.

Again, put $AM = a$, $BE = b$, $DF = c$. The four ⊥s can be expressed in terms of a, b, c. For $BN = a + b$, $DQ = a + c$; and since O is the middle point both of AC and BD, we have $BN + DQ = AM + CP$, each being = twice the ⊥ from O. Hence $(a+b)+(a+c)=a+CP$;

$$\therefore\ CP = (a+b+c).$$

Now, $$BN^2 + DQ^2 - 2AM \,.\, CP = (a+b)^2 + (a+c)^2 - 2a(a+b+c) = b^2 + c^2 = BE^2 + DF^2 = BE^2 + EA^2 = BA^2 = \text{area of square.}$$

Prop. 9.—*If the base* AB *of a △ be divided in* D, *so that* $mAD = nDB$; *then* $mAC^2 + nBC^2 = mAD^2 + nBD^2 + (m+n)CD^2$.



Dem.—Let fall the ⊥ CE; then

$$mAC^2 = m(AD^2 + DC^2 + 2AD \,.\, DE) \qquad \text{(xii.)}$$
$$nBC^2 = n(BD^2 + DC^2 - 2DB \,.\, DE) \qquad \text{(xiii.)}$$

Now, since $m\text{AD} = n\text{DB}$, we have

$$m(2\text{AD}\,.\,\text{DE}) = n(2\text{DB}\,.\,\text{DE}).$$

Hence, by addition, the rectangles disappear, and we get

$$m\text{AC}^2 + n\text{BC}^2 = m\text{AD}^2 + n\text{DB}^2 + (m+n)\,\text{CD}^2.$$

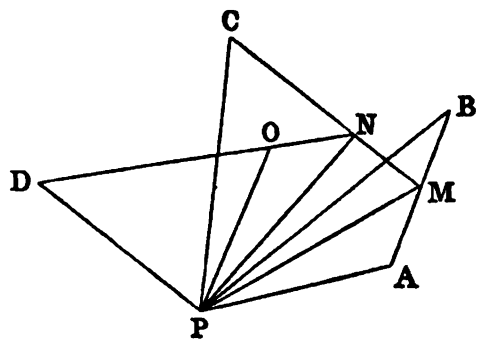
Cor.—If the point D be in the line AB produced, and if $m\text{AD} = n\text{BD}$, we shall have

$$m\text{AC}^2 - n\text{BC}^2 = m\text{AD}^2 - n\text{DB}^2 + (m-n)\,\text{CD}^2.$$

This case is included in the last, if we consider that DB changes sign when the point D passes through B.

Prop. 10.—*If* A, B, C, D, &c., *be any system of n points,* O *their centre of mean position,* P *any other point, the sum of the squares of the distances of the points* A, B, C, D, &c., *from* P, *is greater than the sum of the squares of their distances from* O *by* $n\text{OP}^2$.

Dem.—For the sake of simplicity, let us take four points, A, B, C, D. The method of proof is perfectly general, and can be extended to any number of points. Let M be the middle point of AB; join MC, and divide it in NC, so that MN $= \frac{1}{2}$NC; join ND, and divide in O, so that NO $= \frac{1}{3}$OD, then O is the centre of mean position of the four points A, B, C, D.



Now, applying the theorem of the last article to the several $\triangle$s APB, MPC, NPD, we have

$$\text{AP}^2 + \text{BP}^2 = \text{AM}^2 + \text{MB}^2 + 2\text{MP}^2;$$
$$2\text{MP}^2 + \text{CP}^2 = 2\text{MN}^2 + \text{NC}^2 + 3\text{NP}^2;$$
$$3\text{NP}^2 + \text{DP}^2 = 3\text{NO}^2 + \text{OD}^2 + 4\text{OP}^2.$$

Hence, by addition, and omitting terms that cancel on both sides, we get

$$\text{AP}^2 + \text{BP}^2 + \text{CP}^2 + \text{DP}^2 = \text{AM}^2 + \text{MB}^2 + 2\text{MN}^2 + \text{NC}^2 + 3\text{NO}^2 + \text{OD}^2 + 4\text{OP}^2.$$

Now, if the point P coincide with O, OP vanishes, and we have

$$AO^2 + BO^2 + CO^2 + DO^2 = AM^2 + MB^2 + 2MN^2 + NC^2 + 3NO^2 + OD^2;$$

$$\therefore\ AP^2 + BP^2 + CP^2 + DP^2$$

exceeds $AO^2 + BO^2 + CO^2 + DO^2$ by $4OP^2$.

Cor.—If O be the point of intersection of bisectors of the sides of a $\triangle$, and P any other point; then

$$AP^2 + BP^2 + CP^2 = AO^2 + BO^2 + CO^2 + 3OP^2:$$

for the point of intersection of the bisectors of the sides is the centre of mean position.

Prop. 11.—*The last Proposition may be generalized thus: if* A, B, C, D, &c., *be any system of points,* O *their centre of mean position for any system of multiples* a, b, c, d, &c., *then*

$$a\,.\,AP^2 + b\,.\,BP^2 + c\,.\,CP^2 + d\,.\,DP^2, \text{ \&c.,}$$

exceeds $a\,.\,AO^2 + b\,.\,BO^2 + c\,.\,CO^2 + d\,.\,DO^2$, &c.,

by $(a + b + c + d, \text{ \&c.})OP^2$.

The foregoing proof may evidently be applied to this Proposition. The following is another proof from Townsend's *Modern Geometry*:—

From the points A, B, C, D, &c., let fall $\perp$s AA′, BB′, CC′, DD′, &c., on the line OP; then it is easy to see that O is the centre of mean position for the points A′, B′, C′, D′, and the system of multiples a, b, c, d, &c.

Now we have by Props. xii., xiii., Book II.,

$$AP^2 = AO^2 + OP^2 + 2A'O\,.\,OP;$$
$$BP^2 = BO^2 + OP^2 + 2B'O\,.\,OP;$$
$$CP^2 = CO^2 + OP^2 + 2C'O\,.\,OP;$$
$$DP^2 = DO^2 + OP^2 + 2D'O\,.\,OP, \text{ \&c.};$$

therefore, multiplying by a, b, c, d, and adding, and remembering that

$a\,.\,A'O + b\,.\,B'O + c\,.\,C'O + d\,.\,D'O + \text{\&c.} = 0$ (see I., 18),

we get

$$a \,.\, AP^2 + b \,.\, BP^2 + c \,.\, CP^2 + d \,.\, DP^2, \text{ \&c.,}$$
$$= a \,.\, AO^2 + b \,.\, BO^2 + c \,.\, CO^2 + d \,.\, DO^2 + \text{\&c.,}$$
$$+ (a + b + c + d, \text{ \&c.}) OP^2.$$

This Proposition evidently includes the last.

Cor. 1.—The locus of a point, the sum of the squares of whose distances form any number of given points, multiplied respectively by any system of constants a, b, c, d, is a circle, whose centre is the centre of mean position of the given points for the system of multiples a, b, c, d.

Cor. 2.—The sum of the squares for any system of multiples will be a minimum when the lines are drawn to the centre of mean position.

Prop. 12.—*From the* Propositions vi. and ix. *it follows that, if a line is divided into any two parts, the rectangle of the parts is a maximum, and the sum of their squares is a minimum, when the parts are equal.*

Cor.—If a line be divided into any number of equal parts, the continued product of all the parts is a maximum, and the sum of their squares is a minimum. For if we make any two of the parts unequal, we diminish the continued product, and we increase the sum of the squares.

SECTION II.

Exercises.

1. The second and third Propositions of the Second Book are special cases of the First.

2. Prove the fourth Proposition by the second and third.

3. Prove the sixth by the fifth, and the tenth by the ninth.

4. If the ∠ C of a △ ACB be $\frac{1}{3}$ of two right ∠ s, prove

$$AB^2 = AC^2 + CB^2 - AC \,.\, CB.$$

5. If C be $\frac{2}{3}$ of two right ∠ s, prove

$$AB^2 = AC^2 + CB^2 + AC \,.\, CB.$$

6. In a quad[l]. the sum of the squares of two opposite sides, together with the sum of the squares of the diagonals = sum of the squares of the two remaining sides, together with four times the square of the line joining their middle points.

7. Divide a given line AB in C, so that the rectangle under BC and a given line may be = to the square of AC.

8. Being given the rectangle contained by two lines, and the difference of their squares: construct them.

9. Produce a given line AB to C, so that AC . CB = square of another given line.

10. If a line AB be divided in C so that $AB . BC = AC^2$, prove $AB^2 + BC^2 = 3AC^2$, and $(AB + BC)^2 = 5AC^2$.

11. In the fig. of Prop. xi. prove—

(1). The lines GB, DF, AK, are parallel.

(2). The square of the diameter of the ⊙ about the △ FHK $= 6FK^2$.

(3). The square of the diameter of the ⊙ about the △ FHD $= 6FD^2$.

(4). The square of the diameter of the ⊙ about the △ AHD $= 6AD^2$.

(5). If the lines EB, CH intersect in J, AJ is ⊥ to CH.

12. If ABC be an isosceles △, and DE be ∥ to the base BC, and BE joined, $BE^2 - CE^2 = BC . DE$.

13. If squares be described on the three sides of any △, and the adjacent angular points of the squares joined, the sum of the squares of the three joining lines = three times the sum of the squares of the sides of the △.

14. Given the base AB of a △, both in position and magnitude, and $mAC^2 - nBC^2$: find the locus of C.

15. If from a fixed point P two lines PA, PB, at right angles to each other, cut a given ⊙ in the points A, B, the locus of the middle point of AB is a ⊙.

16. If CD be any line ∥ to the diameter AB of a semicircle, and if P be any point in AB, then

$$CP^2 + PD^2 = AP^2 + PB^2.$$

17. If O be the mean centre of a system of points A, B, C, D, &c., for a system of multiples a, b, c, d, &c.; then, if L and M be any two ∥ lines,

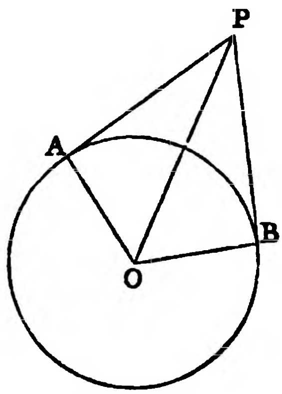
$$\Sigma(a . AL^2) - \Sigma(a . AM^2) = \Sigma(a) . (OL^2 - OM^2).$$

BOOK THIRD.

SECTION I.

Additional Propositions.

Prop. 1.—*The two tangents drawn to a ⊙ from any external point are equal.*

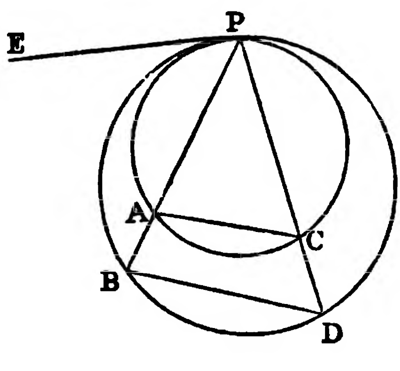


Dem.—Let PA, PB, be the tangents, O the centre of the ⊙. Join OA, OP, OB; then

$$OP^2 = OA^2 + AP^2$$
$$OP^2 = OB^2 + BP^2;$$

but $OA^2 = OB^2$; ∴ $AP^2 = BP^2$, and AP = BP.

Prop. 2.—*If two ⊙s touch at a point* P, *and from* P *any two lines* PAB, PCD *be drawn, cutting the ⊙s in the points* A, B, C, D, *the lines* AC, BD *joining the points of section are parallel.*



Dem.—At P draw the common tangent PE to both ⊙s; then

$\angle$ EPA = $\angle$ PCA (xxxii.)
$\angle$ EPB = $\angle$ PDB.

Hence $\angle$ PCA = $\angle$ PDB, and AC is || to BD (I. xxviii.)

Cor.—If the $\angle$ APC be a right angle, AC and BD will be diameters of the $\odot$s, and then we have the following *important* theorem. The lines drawn from the point of contact of two touching $\odot$s to the extremities of any diameter of one of them will meet the other in points which will be the extremities of a $\parallel$ diameter.

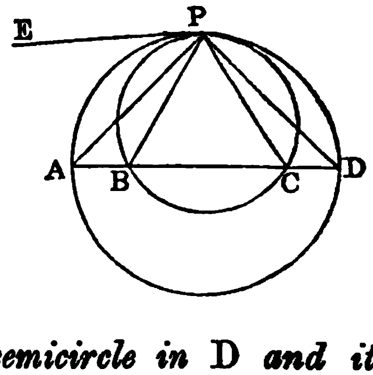
Prop. 3.—*If two $\odot$s touch at* P, *and any line* PAB *cut both $\odot$s in* A *and* B, *the tangents at* A *and* B *are* $\parallel$.

Dem.—Let the tangents at A and B meet the tangents at P in the points E and F.

Now, since AE = EP (1), the $\angle$ APE = $\angle$ PAE. In like manner, the $\angle$ BPF = $\angle$PBF; $\therefore$ the $\angle$ PAE = $\angle$PBF, and AE is $\parallel$ to BF.

This Prop. may be inferred from (2), by supposing the lines PAB, PCD to approach each other indefinitely; then AC and BD will be tangents.

Prop. 4.—*If two $\odot$s touch each other at any point* P, *and any line* A, B, C, D, *cut the $\odot$s in the points* A, B, C, D; *then the* $\angle$ APB = $\angle$ CPD.

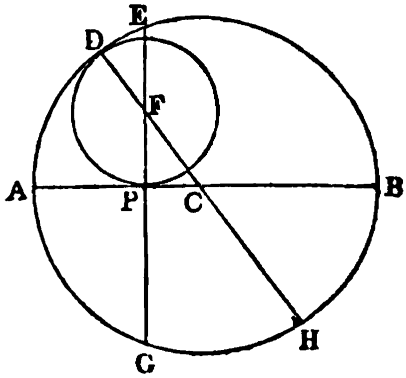


Dem.—Draw a tangent PE at P; then

$\angle$ EPB = $\angle$ PCB (xxxii.)
$\angle$ EPA = $\angle$ PDA.

Hence, by subtraction, $\angle$ APB = $\angle$ CPD.

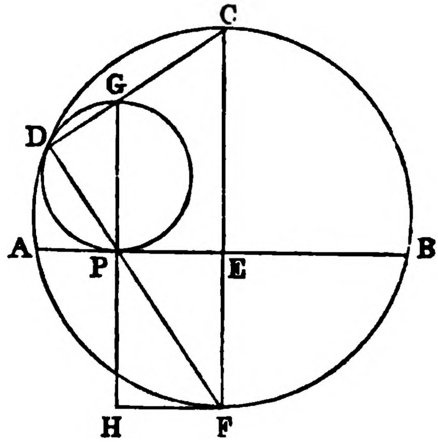
Prop. 5.—*If a $\odot$ touch a semicircle in* D *and its diameter in* P, *and* PE *be* $\perp$ *to the diameter at* P, PE^2 = *twice rectangle contained by the radii of the $\odot$s.*



Dem.—Complete the $\odot$, and produce EP to meet it again in G. Let C and F be the centres; then the line CF will pass through D. Let it meet the outside $\odot$ again in H.

Now, EF . FG = DF . FH (xxxv.), and $PF^2 = DF^2$. Hence, by addition, making use of II., v., EP^2 = DF . DH = twice rectangle contained by the radii.

Prop. 6.—*If a* ⊙ PGD *touch a* ⊙ ABC *in* D *and a chord* AB *in* P, *and if* EF *be* ⊥ *to* AB *at its middle point, and at the side opposite to that of the* ⊙ PGD, *the rectangle contained by* EF *and the diameter of the* ⊙ PGD = *rectangle* AP . PB.



Dem.—Let PG be at right ∠s to AB, then PG is the diameter of the ⊙ PGD. Join DG, DP, and produce them to meet the ⊙ ABC in C and F; then CF is the diameter of the ⊙ ABC, and is ∥ to PG (2); ∴ CF is ⊥ to AB; hence it bisects AB in E (iii.). Through F draw FH ∥ to AB, and produce GP to meet it in H.

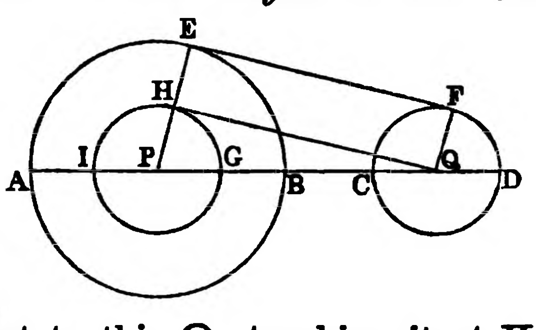
Now, since the ∠s H and D are right ∠s, a semicircle described on GF will pass through the points D and H.

Hence HP . PG = FP . PD = AP . PB; (xxxv.)
but HP = EF; ∴ EF . PG = AP . PB.

This Prop. and its Demonstration will hold true when the ⊙s are external to each other.

Cor.—If AB be the diameter of the ⊙ ABC, this Prop. reduces to the last.

Prop. 7.—*To draw a common tangent to two* ⊙*s.*



Let P be the centre of the greater ⊙, Q the centre of the less, with P as centre, and a radius = to the difference of the radii of the two ⊙s: describe the ⊙ IGH; from Q draw a tangent to this ⊙, touching it at H.

Join PH, and produce it to meet the circumference of the larger ⊙ in E. Draw QF ∥ to PE. Join EF, which will be the common tangent required.

Dem.—The lines HE and QF are, from the construction, =; and since they are ∥, the fig. HEFQ is a ▭; ∴ the ∠PEF = ∠PHQ = right ∠; ∴ EF is a tangent at E; and since ∠ EFQ = ∠ EHQ = right ∠, EF is a tangent at F. The tangent EF is called a *direct* common tangent. If with P as centre, and a radius = to the sum of the two given ⊙s, we had described a ⊙, we should have a common tangent which would pass between the ⊙s, and one which is called a transverse common tangent.

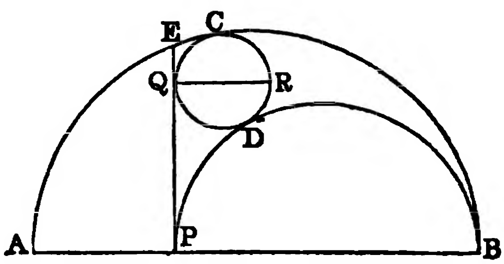
Prop. 8.—*If a line passing through the centres of two ⊙s cut them in the points* A, B, C, D, *respectively, then the square of their direct common tangent is = rectangle* AC . BD.

Dem.—We have (see last fig.) AI = CQ; to each add IC, and we get AC = IQ. In like manner, BD = GQ. Hence AC . BD = IQ . QG = EF^2.

Cor. 1.—If the two ⊙s touch, the square of their common tangent = rectangle contained by their diameters.

Cor. 2.—The square of the transverse common tangent of two ⊙s = AD . BC.

Cor. 3.—If ABC be a semicircle, PE a ⊥ to AB from any point P, CQD a ⊙ touching PE, the semicircle ABC, and the semicircle on PB; then, if QR be the diameter of CQD, AB . QR = EP^2.

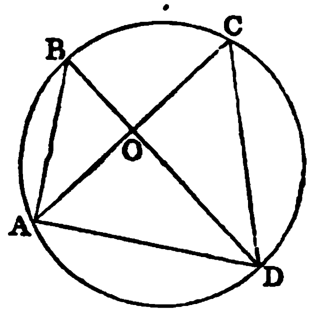


Dem.— $$PB \cdot QR = PQ^2. \qquad (Cor.\ 1)$$
$$AP \cdot QR = EP^2 - PQ^2; \qquad (6)$$
$$\therefore \text{by addition, } AB \cdot QR = EP^2.$$

Cor. 4.—If two ⊙s be described to touch an ordi-

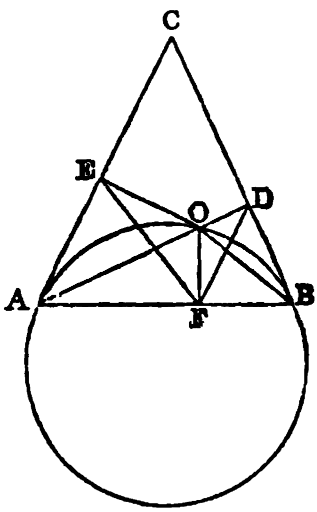
nate of a semicircle, the semicircle itself and the semicircles on the segments of the diameter, they will be = to one another.

Prop. 9.—*In equiangular* △*s the rectangles under the non-corresponding sides about equal* ∠*s are = to one another.*



Dem.—Let the equiangular △s be ABO, CDO, and let them be placed so that the = ∠s at O may be vertically opposite, and that the non-corresponding sides AO, CO may be in one right line, then the non-corresponding sides BO, OD shall be in one right line. Now, since the ∠ ABD = ∠ ACD, the four points A, B, C, D are concyclic (in the circumference of the same ⊙). Hence, the rectangle AO . OC = rectangle BO . OD. (xxxv.)

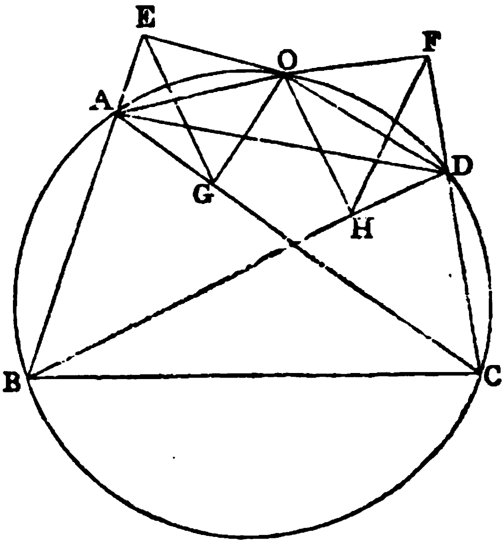
Prop. 10.—*The rectangle contained by the* ⊥*s from any point* O *in the circumference of a* ⊙ *on two tangents* AC, BC, *is = to the square of the* ⊥ *from the same point on their chord of contact* AB.



Dem.—Let the ⊥s be OD, OE, OF. Join OA, OB, EF, DF. Now, since the ∠s ODB, OFD, are right, the quad^l. ODBF is inscribed in a ⊙. In like manner, the quad^l. OEAF is inscribed in a ⊙. Again, since BC is a tangent, the ∠ DBO = ∠ BAO (xxxii.); but ∠ DBO = ∠ DFO (xxi.); and ∠ FAO = ∠ FEO; ∴ ∠ DFO = ∠ FEO. In like manner, ∠ ODF = ∠ EFO; hence the △s ODF, FEO are equiangular, and ∴ the rectangles contained by the non-corresponding sides about the = ∠s DOF, FOE, are equal (9). Hence OD . OE = OF^2.

Prop. 11.—*If from any point* O *in the circumference of a* ⊙ ⊥*s be drawn to the four sides, and to the diagonals of*

an inscribed quad^{l}*., the rectangle contained by the* $\perp$*s on either pair of opposite sides is = to the rectangle contained by the* $\perp$*s on the diagonals.*



Dem.—Let OE, OF, be the $\perp$s on the opposite sides AB, CD; OG, OH, the $\perp$s on the diagonals. Join EF, FH, OA, OD. Now, as in the last Prop., we see that the quad^{ls}. AEOG, DFOH, are inscribed in ⊙s. Hence $\angle$ OEG = $\angle$ OAG, and $\angle$ OHF = $\angle$ ODF. Again, since AODC is a quad^{l}. in a ⊙, the $\angle$ OAC + $\angle$ ODC = two right $\angle$s (xxii.) = $\angle$ ODC + $\angle$ ODF; $\therefore$ the $\angle$ OAC = $\angle$ ODF. Hence the $\angle$ OEG = $\angle$ OHF. In like manner, the $\angle$ OGE = $\angle$ OFH. Hence the $\triangle$s OEG, OFH, are equiangular, and the rectangle OE . OF = rectangle OG . OH.

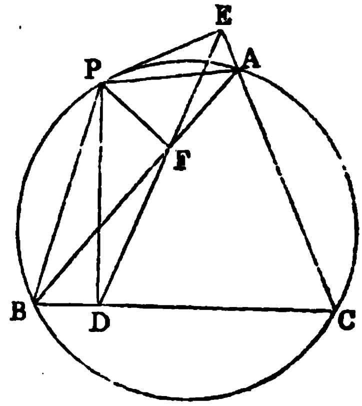
Cor. 1.—The rectangle contained by the $\perp$s on one pair of opposite sides = rectangle contained by the $\perp$s on the other pair of opposite sides. This may be proved directly, or it follows at once from the theorem in the text.

Cor. 2.—If we suppose the points A, B, to become consecutive, and also the points C, D, then AB, CD, become tangents; and from the theorem of this Article we may infer the theorem of Article 10.

Prop. 12.—*The feet* D, E, F *of the three* $\perp$*s let fall on the sides of a* $\triangle$ ABC, *from any point* P *in the circumference of the circumscribed* ⊙ *are collinear.*

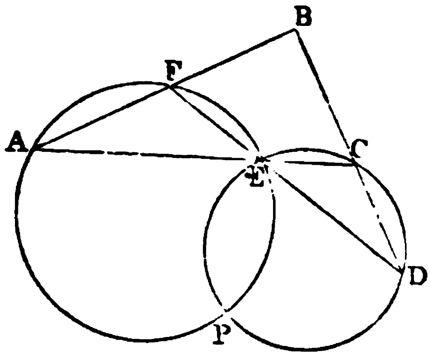
Dem.—Join PA, PB, DF, EF. As in the Demonstrations of the two last Propositions, we see that the quad^{ls}. PBDF, PFAE are inscribed in ⊙s; $\therefore$ the $\angle$s PBD, PFD are = two right $\angle$s (xxii.), and $\angle$s PBD, PAC,

are = two right ∠s (xxii.); ∴ ∠ PFD = ∠ PAC; and since PFAE is a quad[l]. in a ⊙, the ∠ EAP = ∠ EFP; ∴ ∠ PFD + ∠ PFE = ∠ PAC + ∠ PAE = two right ∠s. Hence the points D, F, E, are collinear.



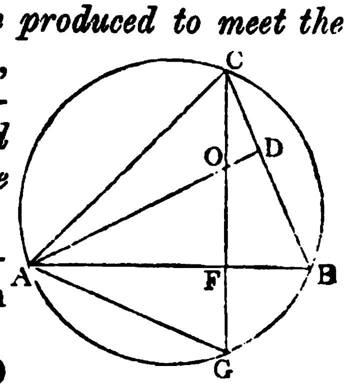
Cor. 1.—If the feet of the ⊥s, drawn from any point P to the sides of a △ ABC be collinear, the locus of P is the ⊙ described about the △.

Cor. 2.—If four lines be given, a point can be found such, that the feet of the four ⊥s from it on the lines will be collinear. For, let the four lines be AB, AC, DB, DF. These lines form four △s. Let the ⊙s described about two of the △s—say AFE, CDE—intersect in P; then it is evident that the feet of the ⊥s from P on the four lines will be collinear.



Cor. 3.—The ⊙s described about the △s ABC, DBF, each passes through the point P. This follows because the feet of the ⊥s from P on the sides of these △s are collinear.

Prop. 13.—*If the ⊥s of a △ be produced to meet the circumference of the circumscribed ⊙, the parts of the ⊥s intercepted between their point of intersection and the circumference are bisected by the sides of the △.*

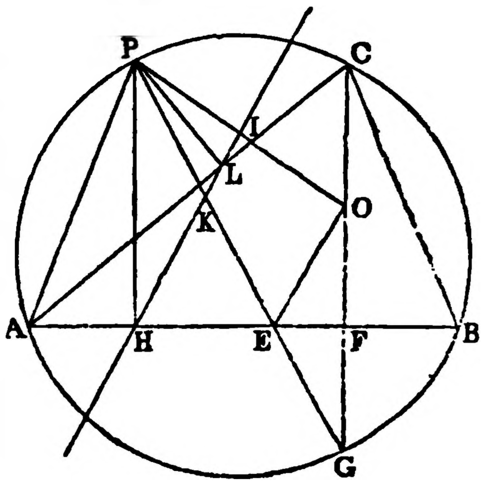


Let AD, CF intersect in O, produce CF to meet the ⊙ in G, then OF = FG.

Dem.—The ∠ AOF = ∠ COD (I. xv.), and ∠ AFO = ∠ CDO, each being right;

$\therefore \angle$ FAO $= \angle$ OCD; but $\angle$ OCD $= \angle$ GAF (xxi.); $\therefore \angle$FAO $= \angle$ FAG, and $\angle$ AFO $= \angle$ AFG, each being right, and AF common. Hence OF = FG.

Prop. 14.—*The line joining any point* P, *in the circumference of a* $\odot$, *to the point of intersection of the* $\perp$*s of an inscribed* $\triangle$, *is bisected by the line of collinearity of the feet of the* $\perp$*s from* P *on the sides of the* $\triangle$.

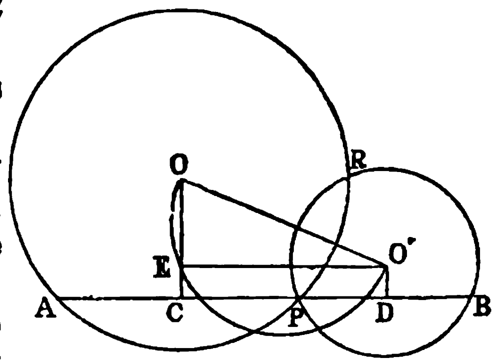


Let P be the point; PH, PL two of the $\perp$s from P on the sides; thus HL is the line of collinearity of the feet of the $\perp$s from P on the sides of the $\triangle$. Let CF be the $\perp$ from C on AB; produce CF to G, and make OF = FG, then O is the point of intersection of the $\perp$s of the $\triangle$. Join OP, intersecting HL in I: it is required to prove that OP is bisected in I.

Dem.—Join AP, PG, and let PG intersect HL in K, and AB in E. Join OE. Now, since APLH is a quad^l. in a $\odot$, the $\angle$ PHK $= \angle$ PAC $= \angle$ PGC $= \angle$ HPK; $\therefore$ PK = KH. Hence KH = KE, and PK = KE. Again, since OF = FG, and FE common, $\angle$ GEF $= \angle$ OEF; but GEF $= \angle$ KEH $= \angle$ KHE; $\therefore \angle$ OEF $= \angle$ KHE; $\therefore$ OE is $\parallel$ to KH; and since EP is bisected in K, OP is bisected in I.

Cor.—If X, Y, Z, W, be the points of intersection of the $\perp$s of the four $\triangle$s AFE, CDE, ABC, DBF (see fig., *Cor.* 2, Prop. 12), then X, Y, Z, W, are collinear. For, let L denote the line of collinearity of the feet of the $\perp$s from P on the sides of the four $\triangle$s. Join PX, PY, PZ, PW. Then, since L joins the points of bisection of the sides of the $\triangle$ PXY, the line XY is $\parallel$ to L. Similarly YZ, ZW, are each $\parallel$ to L. Hence XY, YZ, ZW, form one continuous line.

Prop. 15.—*Through one of the points of intersection of two given ⊙s to draw a line, the sum of whose segments intercepted by the ⊙s shall be a maximum.*



Analysis.—Let the ⊙s intersect in the points P, R, and let APB be any line through P. From O, O′, the centres of the ⊙s, let fall the ⊥s OC, O′D, and draw O′E ∥ to AB. Now, it is evident that AB = 2CD = 2O′E; and that the semicircle described on OO′ as diameter will pass through E. Hence it follows that if AB is a maximum, the chord O′E will coincide with OO′. Therefore AB must be ∥ to the line joining the centres of the ⊙s.

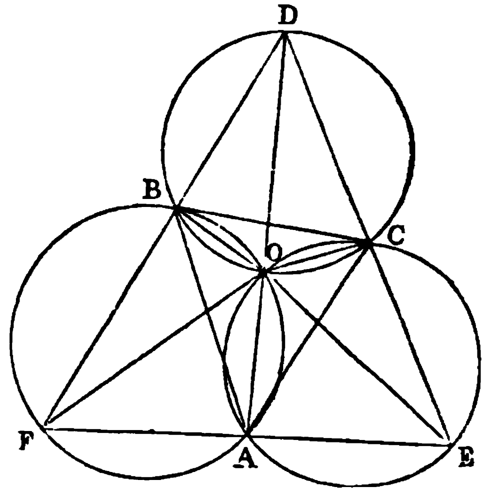
Cor. 1.—If it were required to draw through P a line such that the sum of the segments AP, PB may be = to a given line, we have only to describe a ⊙ from O′ as centre, with a line = half the given line as radius; and where this ⊙ intersects, the ⊙ on OO′ as diameter will determine the point E; and then through P draw a ∥ to O′E.

Def.—*A △ is said to be given in species when its angles are given.*

Prop. 16.—*To describe a △ of given species whose sides shall pass through three given points, and whose area shall be a maximum.*

Analysis.—Let A, B, C be the given points, DEF the required △; then, since ∠ DEF is given in species, the ∠s D, E, F are given, and the lines AB, BC, CA are given by hypotheses; ∴ the ⊙s about the △s ABF, BCD, CAE are given. These three ⊙s will intersect in a common point. For, let the two first intersect in O. Join AO, BO, CO; then ∠ AFB + ∠ AOB = two right ∠s; and ∠ BDC + ∠ BOC = two right ∠s; ∴ the ∠s AFB, BDC, AOB, COB = four right ∠s, and the ∠s

AOB, BOC, COA = four right $\angle$s; $\therefore$ the $\angle$ COA = $\angle$ AFB + $\angle$ BDC: to each add the $\angle$ CEA, and we have the $\angle$ COA + $\angle$ CEA = sum of the three $\angle$s of the $\triangle$ DEF, that is = two right $\angle$s; $\therefore$ the quad[l]. AECO is inscribed in a $\odot$. Hence the three $\odot$s pass through a common point, which is a given point.



Again, since the area of the $\triangle$ DEF is a maximum, each of its sides is a maximum. Hence (15) we have to draw through the point A a line $\parallel$ to the line joining the centres of the $\odot$s ABF, CEA; that is, a line $\perp$ to AO, and join its extremities E, F to the points C, B, respectively.

Cor.—If instead of the maximum $\triangle$ we require to describe a $\triangle$ whose sides will be = to three given lines, the method of solving the question can be inferred from the corollary to the last Prop.

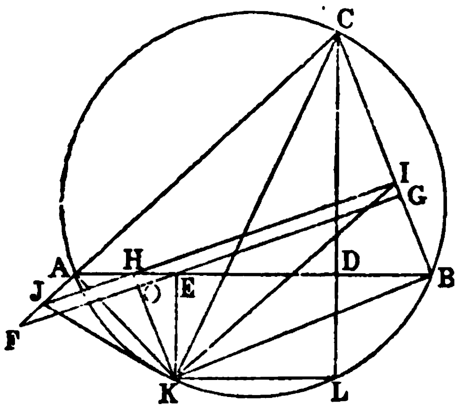
Prop. 17.—*To describe in a given* $\triangle$DEF (see last fig.) *a* $\triangle$ *given in species whose area shall be a minimum.*

Analysis.—Let ABC be the inscribed $\triangle$; describe $\odot$s about the three $\triangle$s ABF, BCD, CAE; then these $\odot$s will have a common point: let it be O. We prove this to be a given point as follows:—The $\angle$ FOE exceeds the $\angle$FDE by the sum of the $\angle$s DFO, DEO; that is, by the sum of the $\angle$s BAO, CAO. Hence the $\angle$ FOE = $\angle$ FDE + $\angle$ BAC; $\therefore$ the $\angle$ FOE is given. In like manner, the $\angle$ EOD is given. Hence the point O will be the point of intersection of two given $\odot$s, and is $\therefore$ given; and, since E and F are given points, the $\angle$ OFE is given; $\therefore$ the $\angle$ OBA is given. In like manner, the $\angle$ OAB is given; $\therefore$ the $\triangle$ OAB is given in species. Now, since the $\triangle$ABC is a minimum, the side AB is a minimum; $\therefore$ OA is a minimum; and

since O is a given point, OA must be ⊥ to EF. Hence the method of inscribing the minimum △ has been found.

Cor.—From the foregoing analysis the method is obvious of inscribing in a given △ another △ whose sides shall be respectively = to three given right lines.

Prop. 18.—*If* ABC *be a* △, *and* CD *a* ⊥ *to* AB; *then if* AE = BD, *it is required to prove that* AB *is the minimum line that can be drawn through* E, *meeting the two fixed lines* AC, BC.

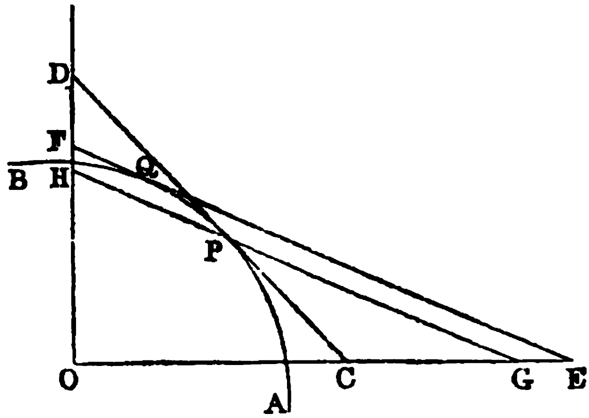


Dem.—Describe a ⊙ about the △ ABC; produce CD to meet it in L, and erect EK ⊥ to AB. Join AK, BK. Through E draw any other line FG; draw KO ⊥ to FG, and produce it to meet AB in H; through H draw JI ∥ to FG. Join JK, IK, CK, KL. Now, since AE = DB, it is evident that EK = DL. Hence KL is ∥ to AB; ∴ the ∠ KLC = ∠ ADC, and is consequently a right ∠; ∴ KC is the diameter of the ⊙; ∴ the ∠ KBC is right, and the ∠ KHI is right; ∴ KHIB is a quad[l]. inscribed in a ⊙; ∴ the ∠ KIH = ∠ KBA. In like manner, the ∠ KJH = ∠ KAB; ∴ the △s IJK and BAK are equiangular; and since IK is greater than KB (the ∠ IBK being right), it follows that IJ is greater than AB; but FG is evidently greater than IJ; ∴ much more is FG greater than AB. Hence AB is the minimum line that can be drawn through E.

Prop. 19.—*If* OC, OD *be any two lines,* AB *any arc of a* ⊙, *or any other curve convex to* O; *then, of all the tangents which can be drawn to* AB, *that whose intercept is bisected at the point of contact cuts off the minimum* △.

Dem.—Let CD be bisected at P, and let EF be any other tangent. Then, through P draw GH ∥ to EF;

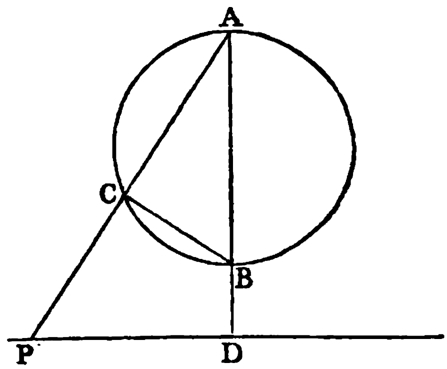
then, since CD is bisected in P, the △ cut off by CD is less than the △ cut off by GH (I. 19); but the △ cut off by GH is less than the △ cut off by EF. Hence the △ cut off by CD is less than the △ cut off by EF



Cor. 1.—Of all △s described about a given ⊙, the equilateral △ is a minimum.

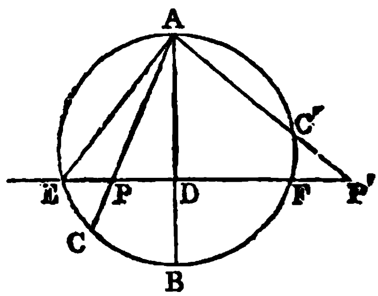
Cor. 2.—Of all polygons having a given number of sides described about a given ⊙, the regular polygon is a minimum.

Prop. 20.—*If* ABC *be a* ⊙, AB *a diameter*, PD *a fixed line* ⊥ *to* AB; *then, if* ACP *be any line cutting the* ⊙ *in* C *and the line* PD *in* P, *the rectangle* AP, AC *is constant.*



Dem.—Since AB is the diameter of the ⊙, the ∠ ACB is right (xxxi.); ∴ the ∠ BCP is right, and the ∠ BDP is right; ∴ the figure BDPC is a quad[l]. inscribed in a ⊙, and, consequently, the rectangle AP . AC = rectangle AB . AD = constant.

Cor. 1.—This Prop. holds true when the line PD cuts the ⊙, as in the diagram; the value of the constant will, in this case, be = AE^2. Hence we have the following:—



Cor. 2.—If A be the middle point of the arc EF, AC any chord cutting the line EF in P, then AP . AC = AE^2.

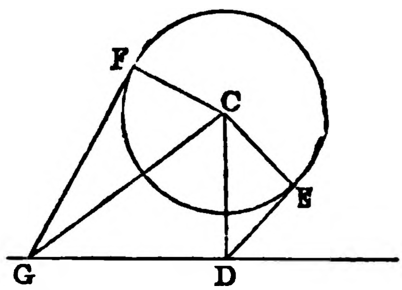
On account of its importance, we shall give an independent proof of this Prop. Thus: join EC, and suppose a ⊙ described about the △ EPC; then the ∠ FEA = ∠ ECA, because they stand on = arcs AF, AE. Hence AE touches the ⊙ EPC (xxxii.); ∴ the rectangle AP . AC = AE^2.

Cor. 3.—If A be a fixed point (see two last figs.), PD a fixed line, and if any variable point P in PD be joined to A, and a point C taken on AP, so that the rectangle AP . AC = constant—say R^2—then, by the converse of this Prop., the locus of the point P is a ⊙.

DEF.—*The point* C *is called the* inverse *of the point* P, *the* ⊙ ABC *the* inverse *of the line* PD, *the fixed point* A *the* centre, *and the constant* R *the* radius *of* inversion.

We shall give more on the subject of inversion in our addition to Book VI.

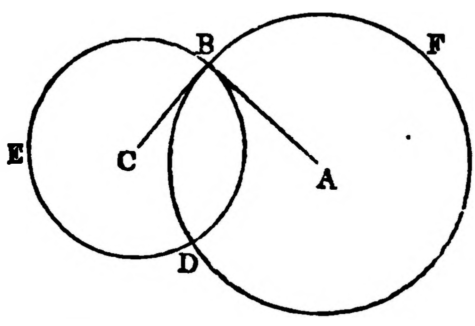
Prop. 21.—*If from the centre of a* ⊙ *a* ⊥ *be let fall on any line* GD, *and from* D *the foot of the* ⊥, *and from any other point* G *in* GD *two tangents* DE, GF *be drawn to the* ⊙, *then* $GF^2 = GD^2 + DE^2$.



Dem.—Let C be the centre of the ⊙. Join CG, CE, CF.

Then $GF^2 = GC^2 - CF^2 = GD^2 + DC^2 - CF^2$
$= GD^2 + DE^2 + EC^2 - CF^2 = GD^2 + DE^2$.

Prop. 22.—*To describe a* ⊙ *having its centre at a given point, and cutting a given* ⊙ *orthogonally* (*at right angles*).



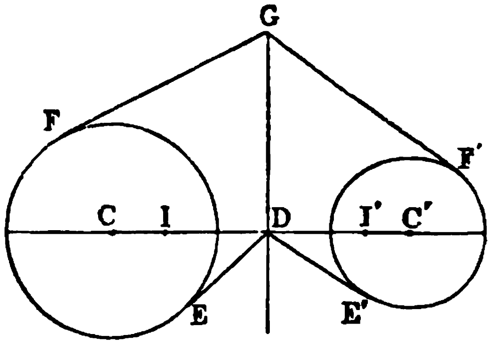
Let A be the given point, BED the given ⊙. From A draw AB, touching the ⊙ BED (xvii.) at B, and from A as centre and AB as radius, describe the ⊙ BFD: this ⊙ will cut BED orthogonally.

Dem.—Let C be the centre of BED. Join CB; then, because AB is a tangent to the ⊙ BED, CB is at right ∠s to AB (xviii.); ∴ CB touches the ⊙ BDF. Now, since AB, CB are tangents to the ⊙s BDE, BDF, these lines coincide with the ⊙s for an indefinitely short distance (a tangent to a ⊙ has two consecutive points common with the ⊙); and, since the lines intersect at right ∠s, the ⊙s cut at right ∠s; that is, orthogonally.

Cor. 1.—The ⊙s cut also orthogonally at D.

Cor. 2.—When two ⊙s cut orthogonally, the square of the distance between their centres = sum of squares of their radii.

Prop. 23.—*If in the line joining the centres of two* ⊙*s a point* D *be found, such that the tangents* DE, DE′ *from it to the* ⊙*s are equal, and if through* D *a line* DG *be drawn* ⊥ *to the line joining the centres, then the tangents from any other point* G *in* DG *to the* ⊙*s will be equal.*



Dem.—Let GF, G′F′ be the tangents. Now, by hypothesis, $DE^2 = DE'^2$. To each add DG^2, and we have

$$GD^2 + DE^2 = GD^2 + DE'^2,$$

or $$GF^2 = GF'^2; \therefore GF = GF'.$$

Def.—*The line* GD *is called the radical axis of the two* ⊙*s; and two points* I, I′, *taken on the line through the centres, so that* DI = DI′ = DE = DE′, *are called the limiting points.*

Cor. 1.—Any circle whose centre is on the radical axis, and which cuts one of the given ⊙s orthogonally, will also cut the other orthogonally, and will pass through the two limiting points.

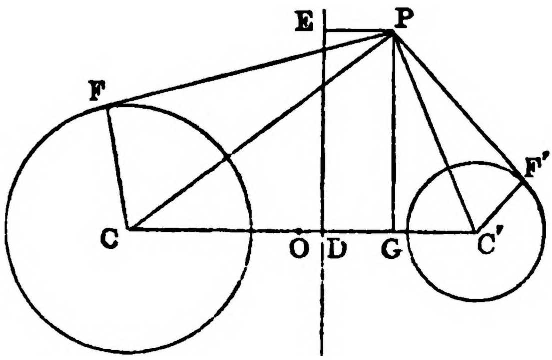
Cor. 2.—If there be a system of three ⊙s, their radical axes taken in pairs are concurrent. For, if

tangents be drawn to the ⊙s from the point of intersection of two of the radical axes, the three tangents will be equal. Hence the third radical axis passes through this point.

DEF.—*The point of concurrence of the three radical axes is called the radical centre of the circles.*

Cor. 3.—The ⊙ whose centre is the radical centre of three given ⊙s, and which cuts one of them orthogonally, cuts the other two orthogonally.

Prop. 24.—*The difference between the squares of the tangents, from any point* P *to two* ⊙*s, is = twice the rectangle contained by the* ⊥ *from* P *on the radical axis and the distance between the centres of the circles.*



Dem.—Let C, C′, be the centres, O the middle point of CC′, DE the radical axis. Let fall the ⊥s PE, PG. Now,

$$CP^2 - C'P^2 = 2CC' . OG \qquad \text{(II., 6)}$$
$$CF^2 - C'F'^2 = CD^2 - C'D^2,$$

because DE is the radical axis

$$= 2CC' . OD.$$

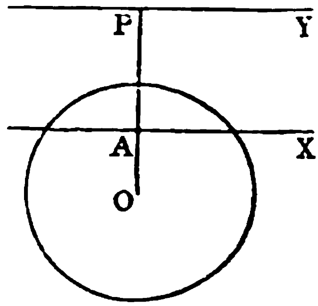
Hence, by subtraction,

$$PF^2 - PF'^2 = 2CC' . DG = 2CC' . EP.$$

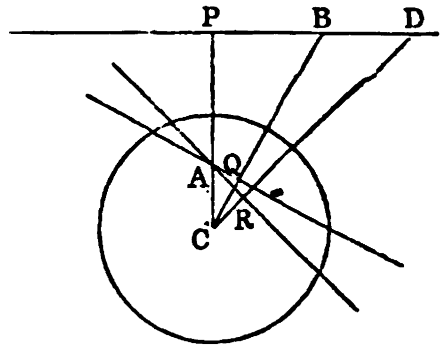
This is the fundamental Prop. in the theory of coaxal circles. For more on this subject, see Section V. Book VI.

DEF.—*If on any radius of a circle two points be taken, one internally and the other externally, so that the rectangle contained by their distances from the centre is = to the square of the radius; then a line drawn* ⊥ *to the*

radius through either point is called the polar *of the other point, which is called, in relation to this* ⊥, *its* pole. *Thus, let* O *be the centre, and let* OA . OP = *radius*²; *then, if* AX, PY *be* ⊥s *to the line* OP, PY *is called the* polar *of* A, *and* A *the* pole *of* PY. *Similarly,* AX *is the polar of* P, *and* P *the pole of* AX.



Prop. 25.—*If* A *and* B *be two points, such that the polar of* A *passes through* B, *then the polar of* B *passes through* A.



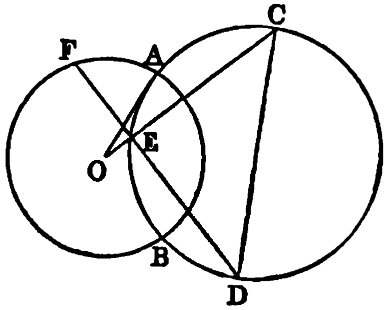
Dem.—Let the polar of A be the line PB; then PB is ⊥ to CP (C being the centre). Join CB, and let fall the ⊥ AQ on CB. Then, since the ∠s P and Q are right ∠s, the quad[l]. ABPQ is inscribed in a ⊙; ∴ CQ . CB = CA . CP = radius²; ∴ AQ is the polar of B.

Cor.—In PB take any other point D. Join CD, and let fall the ⊥ AR on CD. Then AQ, AR are the polars of the points B and D; and we see that the line BD, which joins the points B and D, is the polar of the point A, the intersection of AQ, AR, the polars of B and D. Hence we have the following important theorem:—*The line of connexion of any two points is the polar of the point of intersection of their polars;* or, again: *The point of intersection of any two lines is the pole of the line of connexion of their poles.*

Def.—*Two points, such as* A *and* B, *which possess the property that the polar of either passes through the other, are called* conjugate *points with respect to the* ⊙, *and their polars are called* conjugate *lines.*

Prop. 26.—*If two* ⊙*s cut orthogonally, the extremities*

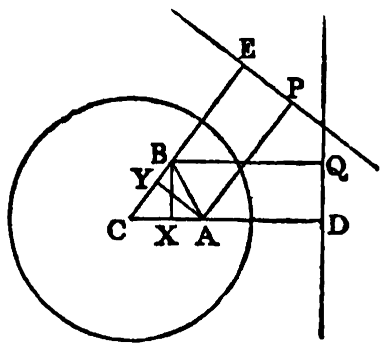
of any diameter of either are conjugate points with respect to the other.



Let the $\odot$s be ABF and CED, cutting orthogonally in the points A, B; let CD be any diameter of the $\odot$ CED; C and D are conjugate points with respect to the $\odot$ ABF.

Dem.—Let O be the centre of the $\odot$ ABF. Join OC, intersecting the $\odot$ CED in E. Join ED, and produce to F. Join OA. Now, because the $\odot$s intersect orthogonally, OA is a tangent to the $\odot$ CED. Hence $OC \cdot OE = OA^2$; that is, $OC \cdot OE$ = square of radius of the $\odot$ ABF; and, since the $\angle$ CED is a right $\angle$, being in a semicircle, the line ED is the polar of C. Hence C and D are conjugate points with respect to the $\odot$ ABF.

Prop. 27.—*If* A *and* B *be two points, and if from* A *we draw a* $\perp$ AP *to the polar of* B, *and from* B *a* $\perp$ BQ *to the polar of* A; *then, if* C *be the centre of the* $\odot$, *the rectangle* $CA \cdot BQ = CB \cdot AP$ (Salmon).



Dem.—Let fall the $\perp$s AY, BX, on the lines CE, CD. Now, since X and Y are right $\angle$s, the semicircle on AB passes through the points X, Y.

$$\therefore CA \cdot CX = CB \cdot CY;$$

and

$$CA \cdot CD = CB \cdot CE,$$

because each = radius^2; $\therefore$ we get, by subtraction,

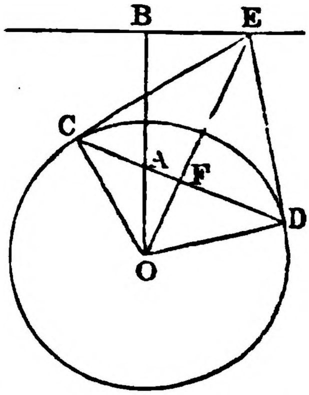
$$CA \cdot DX = CB \cdot EY;$$

or

$$CA \cdot BQ = CB \cdot AP.$$

Prop. 28.—*The locus of the intersection of tangents to*

a ⊙, at the extremities of a chord which passes through a given point, is the polar of the point.



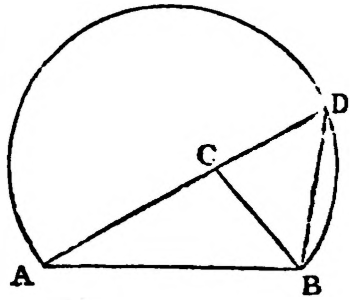
Dem.—Let CD be the chord, A the given point, CE, DE the tangents. Join OA, and let fall the ⊥ EB on OA produced. Join OC, OD. Now, since EC = ED, and EO common, and OC = OD, the ∠ CEO = ∠ DEO. Again, since CE = DE and EF common, and ∠ CEF = ∠ DEF; ∴ the ∠ EFC = ∠EFD. Hence each is right. Now, since the △ OCE is right-angled at C, and CF ⊥ to OE, OF . OE = OC^2; but since the quad[l]. AFEB has the opposite angles B and F right angles, it is inscribed in a ⊙. The rectangle OF . OE = OA . OB; but OF . OE = OC^2; ∴ OA . OB = OC^2 = radius^2; ∴ BE is the polar of A, and this is the locus of the point E.

Cor. 1.—If from every point in a given line tangents be drawn to a given circle, the chord of contact passes through the pole of the given line.

Cor. 2.—If from any given point two tangents be drawn to a given circle, the chord of contact is the polar of the given point.

Prop. 29.—The older geometers devoted much time to the solution of problems which required the construction of △s under certain conditions. Three independent data are required for each problem. We give here a few specimens of the modes of investigation employed in such questions, and we shall give some additional ones under the Sixth Book.

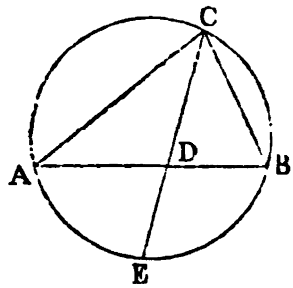
(1). *Given the base of a △, the vertical ∠, and the sum of the sides: construct it.*



Analysis.—Let AB be the △; produce AC to D, and make CD = CB; then AD = sum of sides, and is given, and the △ ADB = half the △ ACB, and is given. *Hence we have the*

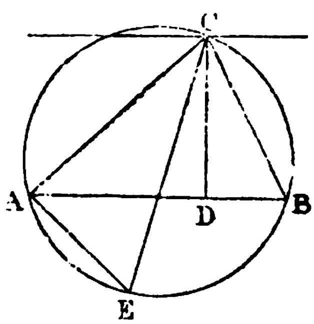
following method of construction:—On the base AB describe a segment of a ⊙ containing an ∠ = half the given vertical ∠, and from the centre A, with a distance = to the sum of the sides as radius, describe a ⊙ cutting this segment in D. Join AD, DB, and make the ∠ DBC = ∠ ADB; then ABC is the △ required.

(2). *Given the vertical ∠ of a △, and the segments into which the line bisecting it divides the base: construct it.*



Analysis.—Let ABC be the △, CD the line bisecting the vertical ∠. Then AD, DB, and the △ ACB are given. Now, since AD, DB are given, AB is given; and since AB and the ∠ ACB are given, the ⊙ ACB is given (xxxiii.); and since CD bisects the ∠ ACB, we have arc AE = EB; ∴ E is a given point, and D is a given point. Hence the line ED is given in position, and ∴ the point C is given.

(3). *Given the base, the vertical ∠, and the rectangle of the sides, construct the △.*



Analysis.—Let ABC be the △; let fall the ⊥ CD; draw the diameter CE; join AE. Now, the ∠ CEA = ∠ CBA (xxi.), and ∠ CAE is right, being in a semicircle (xxxi.); ∴ = ∠ CDB. Hence the △s CAE, CDB are equiangular; ∴ rectangle AC . CB = rectangle CE . CD (9); but rectangle AC . CB is given; ∴ the rectangle CE . CD is given; and since the base and vertical ∠ are given, the ⊙ ACB is given; ∴ the diameter CE is given; ∴ CD is given, and ∴ the line drawn through C ∥ to AB is given in position. Hence the point C is given.

The method of construction is obvious.

NOTE ON PROPOSITION 18.

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If in the fig., Prop. 18, the line BA receive an infinitely small change of position, namely, B along BC, and A along AC; then it is plain the motions of B and A would be the same as if the △ AKB got an infinitely small turn round the point K, which remains fixed: on this account the point K is called the centre of instantaneous rotation for the line AB.

The following proof of Prop. 18 has been communicated to the author by Professor Williamson, F.R.S.:—Through the points A, B draw the lines AM, BM ∥ to BC, AC; then ME is evidently ⊥ to AB; let fall the ⊥ MN on FG; join AG, MG; then the △ FMG is plainly greater than △ AGM; but △ AGM = △ ABM; ∴ △ FMG is greater than △ ABM, and its ⊥ MN is less than ME, the ⊥ of △ AMB; hence the base FG is greater than the base AB.

SECTION II.

Exercises.

1. The line joining the centres of two ⊙s bisects their common chord perpendicularly.

2. If AB, CD be two ∥ chords in a ⊙, the arc AC = BD.

3. If two ⊙s be concentric, all tangents to the inner ⊙ which are terminated by the outer ⊙ are = to one another.

4. If two ⊥s AD, BE of a △ intersect in O, AO . OD = BO . OE.

5. If O be the intersection of the ⊥s of a △, the ⊙s described about the three △s AOB, BOC, COA are equal to one another.

6. If equilateral △s be described on the three sides of any △, the ⊙s described about these equilateral △s pass through a common point.

7. The lines joining the vertices of the original △ to the opposite vertices of the equilateral △ are concurrent.

8. The centres of the three ⊙s in question 6 are the angular points of another equilateral △. This theorem will hold true if

the equilateral △s on the sides of the original △ be turned inwards.

9. The sum of the squares of the sides of the two new equilateral △s in the last question is equal to the sum of the squares of the sides of the original △.

10. Find the locus of the points of bisection of a system of chords which pass through a fixed point.

11. If two chords of a ⊙ intersect at right angles, the sum of the squares of their four segments = square of the diameter.

12. If from any fixed point C a line CD be drawn to any point D in the circumference of a given ⊙, and a line DE be drawn ⊥ to CD meeting the ⊙, again in E, the line EF drawn through E ∥ to CD will pass through a fixed point.

13. Given the base of a △ and the vertical ∠, prove that the sum of the squares of the sides is a maximum or a minimum when the △ is isosceles, according as the vertical ∠ is acute or obtuse.

14. Describe the maximum rectangle in a given segment of a ⊙.

15. Through a given point inside a ⊙ draw a chord which shall be divided as in Euclid, Prop. XI., Book II.

16. Given the base of a △ and the vertical ∠, what is the locus—(1) of the intersection of the ⊥s; (2) of the bisectors of the base ∠s?

17. Of all △s inscribed in a given ⊙, the equilateral △ is a maximum.

18. The square of the third diagonal of a quad[l]. inscribed in a ⊙ is = to the sum of the squares of tangents to the ⊙ from its extremities.

19. The ⊙, whose diameter is the third diagonal of a quad[l]. inscribed in another ⊙, cuts the latter orthogonally.

20. If from any point in the circumference of a ⊙ three lines be drawn to the angular points of an inscribed equilateral △, one of these lines is = to the sum of the other two.

21. If the feet of the ⊥s of a △ be joined, the △ thus formed will have its angles bisected by the ⊥s of the original △.

22. If all the sides of a quad[l]. or polygon, except one, be given in magnitude and order, the area will be a maximum, when the remaining side is the diameter of a semicircle passing through all the vertices.

23. The area will be the same in whatever order the sides are placed.

24. If two quad^{ls}. or polygons be equilateral, and if one be inscribed in a ⊙, it will be greater than the other.

25. If from any point P without a ⊙ a secant be drawn cutting the ⊙ in the points A, B, then if C be the middle point of the polar of P, the ∠ ACB is bisected by the polar of P.

26. If OPP′ be any line cutting a ⊙, J, in the points PP′, then if two ⊙s passing through O touch J in the points P, P′ respectively, the difference between their diameters is = to the diameter of J.

27. Given the base, the difference of the base ∠s, and the sum or difference of the sides of a △, construct it.

28. Given the base, the vertical ∠, and the bisector of the vertical ∠ of a △, construct it.

29. Draw a right line through the point of intersection of two ⊙s, so that the sum or the difference of the squares of the intercepted segments shall be given.

30. If an arc of a ⊙ be divided into two equal, and into two unequal parts, the rectangle contained by the chords of the unequal parts, together with the square of the chord of the arc between the points of section, is equal to the square of the chord of half the arc.

31. If A, B, C, D be four points, ranged in order on a straight line, find on the same line a point O, such that the rectangle OA . OD shall = the rectangle OB . OC.

32. In the same case find the locus of a point P if the ∠ APB = ∠ CPD.

33. Given two points A, B, and a ⊙ X, find in X a point C, so that the ∠ ACB may be either a maximum or a minimum.

34. The bisectors of the ∠s, at the extremities of the third diagonal of a quad^{l}. are ⊥ to each other.

35. If the base and the sum of the sides of a △ be given, the rectangle contained by the ⊥s from the extremities of the base on the bisector of the external vertical ∠ is given.

36. If any hexagon be inscribed in a ⊙, the sum of three alternate ∠s = sum of the three remaining angles.

37. A line of given length MN slides between two fixed lines OM, ON; then, if MP, NP be ⊥ to OM, ON, the locus of P is a ⊙.

38. State the theorem corresponding to 35 for the internal bisector of the vertical ∠.

39. If AB, AC, AD be two adjacent sides and the diagonal of a ▭, and if a ⊙ passing through A cut these lines in the points P, Q, R, then

$$AB . AP + AC . AQ = AD . AR.$$

40. Draw a chord CD of a semicircle ∥ to a diameter AB, so as to subtend a right ∠ at a given point P in AB (See Exercise 16, Book II).

41. Find a point in the circumference of a given ⊙, such that the lines joining it to two fixed points in the circumference may make a given intercept on a given chord of the ⊙.

42. In a given ⊙ describe a △ whose three sides shall pass through three given points.

43. If through any point O three lines be drawn respectively ∥ to the three sides of a △ intersecting the sides in the points A, A′, B, B′, C, C,′ then the sum of the rectangles AO . OA′, BO . OB′, CO . OC′ is = to the rectangle contained by the segments of the chord of the circumscribed ⊙ which passes through O.

44. The lines drawn from the centre of the circle described about a △ to the angular points are ⊥ to the sides of the △ formed by joining the feet of the ⊥s of the original △.

45. If a ⊙ touch a semicircle and two ordinates to its diameter, the rectangle under the remote segments of the diameter is = to the square of the ⊥ from the centre of the ⊙ on the diameter of the semicircle.

46. If AB be the diameter of a semicircle, and AC, BD two chords intersecting in O, the ⊙ about the △ OCD intersects the semicircle orthogonally.

47. If the sum or difference of the tangents from a variable point to two ⊙s be equal to the part of the common tangent of the two ⊙s between the points of contact, the locus of the point is a right line.

48. If pairs of common tangents be drawn to three ⊙s, and if one triad of common tangents be concurrent, the other triad will also be concurrent.

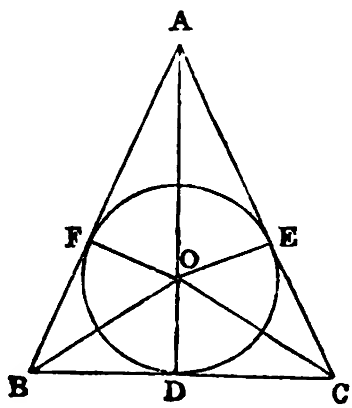
BOOK FOURTH.

SECTION I.

Additional Propositions.

Prop. 1.—*If a ⊙ be inscribed in a △, the distance from the angular points of the △ to the points of contact on the sides are respectively = to the remainders that are left, when the lengths of the sides are taken separately from their half sum.*

Dem.—Let ABC be the △, D, E, F, the points of contact. Now, since the tangents from an external point are equal, we have AE = AF, BD = BF, CD = CE. Hence AE + BC = AB + CE = half sum of the three sides BC, CA, AB; and denoting these sides by the letters a, b, c, respectively, and half their sum by s, we have



$$AE + a = s;$$

$$\therefore AE = s - a.$$

In like manner $\quad BD = s - b; \; CE = s - c.$

Cor. 1.—If r denote the radius of the inscribed ⊙, the area of the △ = rs.

For, let O be the centre of the inscribed ⊙, then we have

$$BC \,.\, r = 2 \triangle BOC$$

$$CA \,.\, r = 2 \triangle COA$$

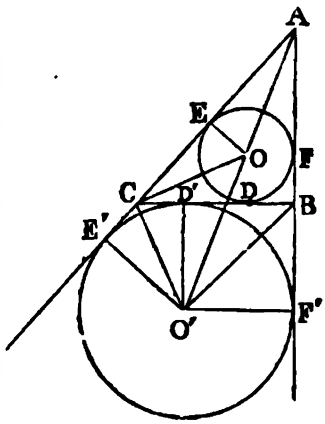
$$AB \,.\, r = 2 \triangle AOB;$$

$$\therefore (B + CA + AB)r = 2 \triangle ABC;$$

that is, $$2sr = 2 \triangle ABC;$$

$$\therefore sr = \triangle ABC. \qquad (\alpha)$$

Cor. 2.—If the ⊙ touch the side BC externally, and the sides AB, AC produced; that is, if it be an escribed ⊙, and if the points of contact be denoted by D′, E′, F′, it may be proved in the same manner that AE′ = AF′ = s; BD′ = BF′ = $s - c$; CD′ = CE′ = $s - b$.



These Propositions, though simple, are very important.

Cor. 3.—If r' denote the radius of the escribed ⊙, which touches the side BC (a) externally,

$$r'(s - a) = \triangle ABC.$$

Dem.— $$E'O' \,.\, AC = 2 \triangle AO'C;$$

that is $$r' \,.\, b = 2 \triangle AO'C.$$

In like manner $$r' \,.\, c = 2 \triangle AO'B,$$

and $$r' \,.\, a = 2 \triangle BO'C.$$

Hence $$r'(b + c - a) = 2 \triangle ABC;$$

that is $$r' \,.\, 2(s - a) = 2 \triangle ABC;$$

$$\therefore r' \,.\, (s - a) = \triangle ABC. \qquad (\beta)$$

Cor. 4.—The rectangle $r \,.\, r' = (s - b)(s - c)$.

Dem.—Since CO bisects the ∠ ACB, and CO′ bisects the ∠ BCE′, CO is at right ∠s to CO′; ∴ the ∠ ECO + ∠ E′CO′ = a right ∠; and ∠ ECO + ∠ COE = one right ∠; ∴ ∠ E′CO′ = ∠ COE. Hence the △s E′CO′, EOC are equiangular; and

$$\therefore E'O' \,.\, EO = E'C \,.\, CE; \qquad \text{(III. 9.)}$$

that is $$r \,.\, r' = (s - b)(s - c). \qquad (\gamma)$$

Cor. 5.—If we denote the area of the △ ABC by N, we shall have

$$\mathrm{N} = \sqrt{s,\ (s-a),\ (s-b),\ (s-c)}.$$

For, by equations (α) and (β), we have

$$rs = \mathrm{N}, \text{ and } r'(s-a) = \mathrm{N}.$$

Therefore, multiplying and substituting from (γ), we get

$$\mathrm{N}^2 = s(s-a)(s-b)(s-c);$$

$$\therefore\ \mathrm{N} = \sqrt{s(s-a)(s-b)(s-c)}.$$

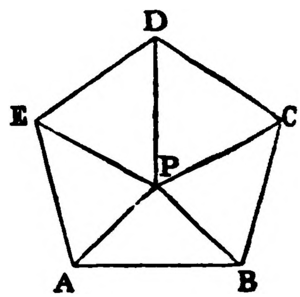
Cor. 6.— $\mathrm{N} = \sqrt{r\,.\,r'\,.\,r''\,.\,r'''}$;

where r'', r''' denote the radii of the escribed circles, which touch the sides b, c, externally.

Cor. 7.—If the △ ABC be right-angled, having the ∠ C right,

$$r = s - c;\ r' = s - a;\ r'' = s - b;\ r''' = s.$$

Prop 2.—*If from any point ⊥s be let fall on the sides of a regular polygon of n sides, their sum = n times the radius of the inscribed* ⊙.



Dem.—Let the given polygon be, say a pentagon ABCDE, and P the given point, and the ⊥s from P on the sides AB, BC, &c., be denoted by p_1, p_2, p_3, &c., and let the common lengths of the sides of the polygon be s, then

$$2\,\triangle\,\mathrm{APB} = sp_1;$$
$$2\,\triangle\,\mathrm{BPC} = sp_2;$$
$$2\,\triangle\,\mathrm{CPD} = sp_3;$$
&c., &c.;

∴ by addition, twice the pentagon

$$= s\,(p_1 + p_2 + p_3 + p_4 + p_5).$$

Again, if we suppose O to be the centre of the inscribed ⊙, and R its radius, we get, evidently,

$$2\,\triangle\,\mathrm{AOB} = \mathrm{R}s;$$

but the pentagon = 5 $\triangle$ AOB;

$$\therefore 2 \text{ pentagon} = 5Rs;$$

$$\therefore s\,(p_1+p_2+p_3+p_4+p_5) = 5Rs.$$

Hence $$p_1+p_2+p_3+p_4+p_5 = 5R.$$

Prop. 3.—*If a regular polygon of n sides be described about a* $\odot$, *the sum of the* $\perp$*s from the points of contact on any tangent to the* $\odot$ = nR.

Dem.—Let A, B, C, D, E, &c., be the points of contact of the sides of the polygon with the $\odot$, L any tangent to the $\odot$, and P its point of contact. Now, the $\perp$s from the points A, B, C, &c., on L, are respectively = to the $\perp$s from P on the tangents at the same points; but the sum of the $\perp$s from P on the tangents at the points A, B, C, &c., = nR (2). Hence the sum of the $\perp$s from the points A, B, C, &c., on L = nR.

Cor. 1.—The sum of the $\perp$s from the angular points of an inscribed polygon of n sides = n times the $\perp$ from the centre on the same line.

Cor. 2.—The centre of mean position of the angular points of a regular polygon is the centre of its circumscribed $\odot$.

For, since there are n points, the sum of the $\perp$s from these points on any line = n times the $\perp$ from their centre of mean position on the line (I., 17); $\therefore$ the $\perp$ from the centre of the circumscribed $\odot$ on any line is = to the $\perp$ from the centre of mean position on the same line; and, consequently, these centres must coincide.

Cor. 3.—The sum of the $\perp$s from the angular points of an inscribed polygon on any diameter is zero; or, in other words, the sum of the $\perp$s on one side of the diameter = sum of the $\perp$s on the other side.

Prop. 4.—*If a regular polygon of n sides be inscribed in a* $\odot$, *whose radius is* R, *and if* P *be any point whose distance from the centre of the* $\odot$ *is* R′, *then the sum of the squares of all the lines from* P *to the angular points of the polygon* = $n\,(R^2 + R'^2)$.

Dem.—Let O be the centre of the ⊙, then O is the mean centre of the angular points, hence (II., 10) the sum of the squares of the lines drawn from P to the angular points exceeds the sum of the squares of the lines drawn from O by $n\text{OP}^2$, that is by $n\text{R}'^2$; but all the lines drawn from O are = to one another, each being the radius. Hence the sum of their squares is $n\text{R}^2$. Hence the Proposition is proved.

Cor. 1.—If the point P be in the circumference of the ⊙, we have the following theorem:—*The sum of the squares of the lines drawn from any point in the circumference of a* ⊙ *to the angular points of an inscribed polygon* $= 2n\text{R}^2$.

The following is an independent proof of this theorem:—It is seen at once, if we denote the ⊥s from the angular points on the tangent at P by p_1, p_2, &c., that

$$2\text{R}\,.\,p_1 = \text{AP}^2;$$
$$2\text{R}\,.\,p_2 = \text{BP}^2;$$
$$2\text{R}\,.\,p_3 = \text{CP}^2, \text{ \&c.}$$

Hence

$$2\text{R}\,(p_1 + p_2 + p_3 + \text{\&c.}) = \text{AP}^2 + \text{BP}^2 + \text{CP}^2, \text{ \&c.};$$

or

$$2\text{R}\,.\,n\text{R} = \text{AP}^2 + \text{BP}^2 + \text{CP}^2, \text{ \&c};$$

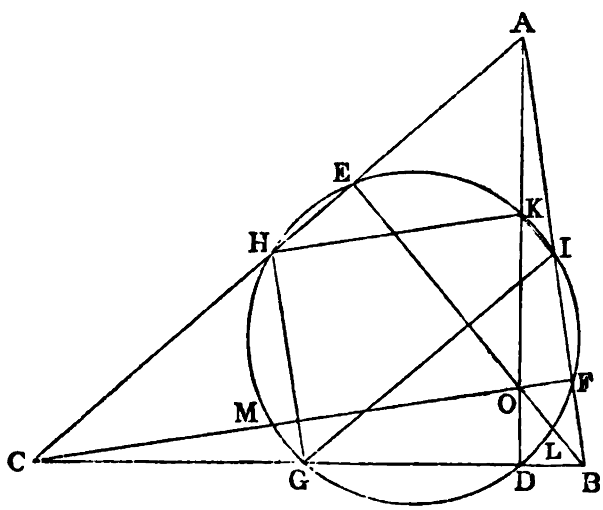
∴ the sum of the squares of all the lines from P $= 2n\text{R}^2$.

Cor. 2.—The sum of the squares of all the diagonals of a regular polygon of n sides, inscribed in a ⊙ whose radius is R, is $n^2\text{R}^2$.

This follows from supposing the point P to coincide with each angular point in succession, and adding all the results, and taking half, because each line occurs twice.

Prop. 5.—*If* O *be the point of intersection of the three* ⊥*s* AD, BE, CF *of a* △ ABC, *and if* G, H, I *be the middle points of the sides of the* △, *and* K, L, M *the middle points of the lines* OA, OB, OC; *then the nine points* D, E, F; G, H, I; K, L, M, *are in the circumference of a* ⊙.

Dem.—Join HK, HG, IK, IG; then, because AG is bisected in K, and AC in H, HK is ∥ to CO. In like manner, HG is ∥ to AB. Hence the ∠ GHK is = to the ∠ between CO and AB; ∴ it is a right ∠; consequently, the ⊙ described on GK as diameter passes through H. In like manner, it passes through I; and since the ∠ KDG is right, it passes through D; ∴ the



⊙ through the three points G, H, I, passes through the two points D, K. In like manner, it may be proved that it passes through the pairs of points E, L, F, M. Hence it passes through the nine points. The ⊙ through the middle points of the sides of a △ is called, on account of the property we have just proved, "The *Nine-points Circle* of the Triangle."

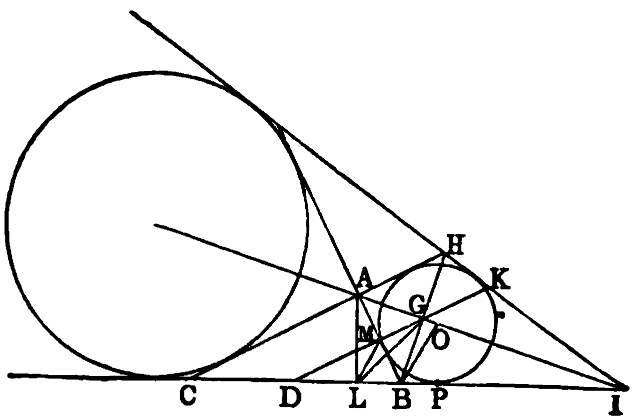
Prop. 6.—*To draw the fourth common tangent to the two escribed ⊙s of a plane △ which touch the base produced, without describing those ⊙s.*

Con.—From B, one of the extremities of the base, let fall a ⊥ BG on the external bisector AI of the vertical ∠ of the △ABC; produce BG and AI to meet the sides CA, CB of the △ in the points H and I; then the line joining the points H and I is the fourth common tangent.

Dem.—The △s BGA, HGA have the side AG com-

mon, and the ∠s adjacent to this side in the two △s = each to each; hence AH = AB. Again, the △s AHI, ABI have the sides AH, AI; and the included ∠ in the one = to the two sides AB, AI and the included ∠ in the other; ∴ the ∠ HIA = ∠ BIA.

Now, bisect the ∠ ABI by the line BO, and it is evident, by letting fall ⊥s on the four sides of the quad[l]. ABIH from the point O, that the four ⊥s are = to one another. Hence the ⊙, having O as centre, and any of these ⊥s as radius, will be inscribed in the quad[l].; ∴ HI is a tangent to the escribed ⊙, which touches AB externally. In like manner, it may be proved that HI touches the escribed ⊙, which touches AC externally. Hence HI is the fourth common tangent to these two ⊙s.



Cor. 1.—If D be the middle point of the base BC, the ⊙, whose centre is D and whose radius is DG, is orthogonal to the two escribed ⊙s which touch BC produced.

For, let P be the point of contact of the escribed ⊙, which touches AB externally, then

$$PD = CP - CD = \tfrac{1}{2}(a + b + c) - \tfrac{1}{2}a = \tfrac{1}{2}(b + c);$$

and since BH is bisected in G, and BC in D,

$$DG = \tfrac{1}{2}CH = \tfrac{1}{2}(AB + AC) = \tfrac{1}{2}(b + c);$$

∴ the ⊙, whose centre is D and radius DG, will cut the ⊙ which touches at P orthogonally.

Cor. 2.—Let DG cut AB in M, and HI in K, and from A let fall the ⊥ AL, then the quad^l. LMKI is inscribed in a ⊙.

For, since the ∠s ALB, AGB are right, ALBG is a quad^l. in a ⊙, and M is the centre of the ⊙; ∴ ML = MB, and ∠ MLB = ∠ MBL. Again, ∠ MKI = ∠ AHI = ∠ ABI; ∴ ∠ MKI + ∠ MLI = ∠ ABI + ∠ MBL = two right ∠s. Hence MKIL is a quad^l. inscribed in a ⊙.

Prop. 7.—*The "Nine-points Circle" is the inverse of the fourth common tangent to the two escribed ⊙s, which touch the base produced, with respect to the ⊙ whose centre is at the middle point of the base, and which cuts these ⊙s orthogonally.*

Dem.—The ∠ DML (see fig., last Prop.) = twice ∠ DGL (III. xx.); and the ∠ HIL = twice ∠ AIL; but ∠ DML = ∠ HIL, since MKIL is a quad^l. in a ⊙; ∴ the ∠ DGL = ∠ GIL. Hence, if a ⊙ be described about the △ GIL it will touch the line GD (III. xxxii.); ∴ DL . DI = DG^2; ∴ the point L is the inverse of the point I, with respect to the ⊙ whose centre is D and radius DG. Again, since MKIL is a quad^l. in a ⊙, DM . DK = DL . DI, and, ∴ = DG^2. Hence the point M is the inverse of K, and ∴ the ⊙ described through the points DLM is the inverse of the line HI (III. 20); that is, the "Nine-points Circle" is the inverse of the fourth common tangent, with respect to the ⊙ whose centre is the middle point of the base, and whose radius is = to half sum of the two remaining sides.

Cor. 1.—In like manner, it may be proved that the "Nine-points Circle" is the inverse of the fourth common tangent to the inscribed ⊙ and the escribed ⊙, which touches the base externally, with respect to the ⊙ whose centre is the middle point of the base, and whose radius is = half difference of the remaining sides.

Cor. 2.—The "Nine-points Circle" touches the inscribed and the escribed ⊙s of the △.

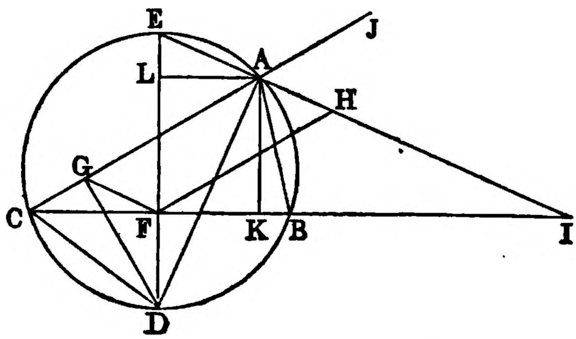
For, since it is the inverse of the fourth common

tangent to the two escribed ⊙s which touch the base produced, with respect to the ⊙ whose centre is D, and which cuts these ⊙s orthogonally; if we join D to the points of contact of the fourth common tangent, the points where the joining lines meet these ⊙s again will be the inverses of the points of contact. Hence they will be common both to the "Nine-points Circle" and the escribed ⊙s; ∴ the "Nine-points Circle" touches these escribed ⊙s; and in a similar way the points of contact with the inscribed ⊙ and escribed ⊙ which touch the base externally may be found.

Cor. 3.—Since the "Nine-points Circle" of a plane △ is also the "Nine-points Circle" of each of the three △s into which it is divided by the lines drawn from the intersection of its ⊥s to the angular points, we see that the "Nine-points Circle" touches also the inscribed and escribed ⊙s of each of these △s.

Prop. 8.—The following Propositions, in connexion with the ⊙ described about a △, are very important.

(1). The lines which join the extremities of the diameter, which is ⊥ to the base of a △ to the vertical ∠, are the internal and external bisectors of the vertical ∠.



Dem.—Let DE be the diameter ⊥ to BC. Join AD, AE. Now, from the construction, we have the arc CD = arc BD. Hence the ∠ CAD = ∠ DAB; ∴ AD is the internal bisector of the ∠ CAB. Again, since DE is the diameter of the ⊙, the ∠ DAE is right; ∴ the ∠ DAE = ∠ DAH; and from these, taking

away the equal $\angle$s CAD, BAD, we have the $\angle$ CAE $= \angle$ BAH; $\therefore$ $\angle$ JAH $= \angle$ BAH. Hence AH is the external bisector.

(2). If from D a $\perp$ be let fall on AC, the segments AG, GC into which it divides AC are respectively the half sum and the half difference of the sides AB, AC.

Dem.—Join CD, GF. Draw FH $\parallel$ to AC. Since the $\angle$s CGD, CFD are right, the figure CGFD is a quad[l]. in a $\odot$. Hence the $\angle$ AGF $= \angle$ CDE (III., xxii.) $= \angle$ CAE (III., xxi.); $\therefore$ GF is $\parallel$ to AE. Hence AHFG is a $\square$; and AG $=$ FH $= \frac{1}{2}$ sum of AB, AC (I., 11, *Cor.* 1). Again, GC $=$ AC $-$ AG $=$ AC $- \frac{1}{2}$(AB $+$ BC) $= \frac{1}{2}$(AC $-$ AB).

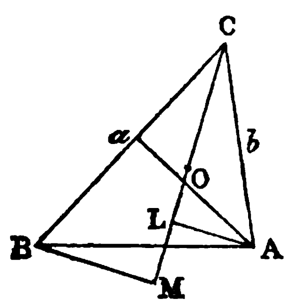
(3). If from E a $\perp$ EG$'$ be drawn to AC, CG$'$, and AG$'$ are respectively the half sum and the half difference of AC, AB. This may be proved like the last.

(4). Through A draw AL $\perp$ to DE. The rectangl DL . EF $=$ square of half sum of the sides AC, AB.

Dem.—The $\triangle$s ALD, EFI have evidently the $\angle$s at D and I equal, and the right $\angle$s at L and F are equal. Hence the $\triangle$s are equiangular; $\therefore$ DL . EF $=$ AL . FI $=$ FK . FI $=$ square of half sum of the sides (Prop. 7).

(5). In like manner it may be proved that EL . FD $=$ square of half difference of AC, AB.

Prop. 9.—*If a, b, c denote, as in* Prop. 1, *the lengths of the sides of the* $\triangle$ ABC, *then the centre of the inscribed* $\odot$ *will be the centre of mean position of its angular points for the system of multiples a, b, c.*



Dem.—Let O be the centre of the inscribed $\odot$. Join CO; and on CO produced let fall the $\perp$s AL, BM. Now, the $\angle$s ACL, BCM have the $\angle$ ACL $= \angle$ BCM; and the $\angle$ ALC $=$ BMC. Hence they are equiangular;

$$\therefore \text{BC . AL} = \text{AC . BM}; \qquad \text{(III. 9)}$$

or
$$a \,.\, \text{AL} = b \,.\, \text{BM}. \qquad (a)$$

Now, if we introduce the signs + and −, since the ⊥s AL, BM fall on different sides of CL, they must be affected with contrary signs; ∴ the equation (α) expresses that *a* times the ⊥ from A on CO + *b* times the ⊥ from B on CO = 0; and since the ⊥ from C on CO is evidently = 0, we have the sum of

a times ⊥ from A,
b times ⊥ from B,
c times ⊥ from C, on CO = 0.

Hence the line CO passes through the centre of mean position for the system of multiples *a*, *b*, *c*. In like manner, AO passes through the centre of mean position. And since a point which lies on each of two lines must be their point of intersection, O must be the centre of mean position for the system of multiples *a*, *b*, *c*.

Cor. 1.—If O′, O″, O‴ be the centres of the escribed ⊙s, O′ is the centre of mean position for the system of multiples − *a*, + *b*, + *c*; O″ for the system + *a*, − *b*, + *c*; and O‴ for the system + *a*, + *b*, − *c*.

SECTION II.

Exercises.

1. The square of the side of an equilateral △ inscribed in a ⊙ = three times the square of the radius.

2. The square described about a ⊙ = twice the inscribed square.

3. The inscribed hexagon = twice the inscribed equilateral △.

4. In the construction of IV., x., if F be the second point in which the ⊙ ACD intersects the ⊙ BDE, and if we join AF, DF, the △ ADF has each of its base ∠s double the vertical ∠. The same property holds for the △s ACF, BCD.

5. The square of the side of a hexagon inscribed in a ⊙, together with the square of the side of a decagon = square of the side of a pentagon.

6. Any diagonal of a pentagon is divided by a consecutive diagonal into two parts, such that the rectangle contained by the whole and one part is = to the square of the other part.

7. Divide an ∠ of an equilateral △ into five = parts.

8. Inscribe a ⊙ in a given sector of a ⊙.

9. The locus of the centre of the ⊙ inscribed in a △, whose base and vertical ∠ are given, is a ⊙.

10. If tangents be drawn to a ⊙ at the angular points of an inscribed regular polygon of any number of sides, they will form a circumscribed regular polygon.

11. The line joining the centres of the inscribed and circumscribed ⊙s subtends at any of the angular points an ∠ equal to half the difference of remaining ∠s.

12. Inscribe an equilateral △ in a given square.

13. The six lines of connexion of the centres of the inscribed and escribed ⊙s of a plane △ are bisected by the circumference of a circumscribed ⊙.

14. Describe a regular octagon in a given square.

15. A regular polygon of any number of sides has one ⊙ inscribed in it, and another circumscribed about it, and the two ⊙s are concentric.

16. If O′, O″, O,‴ be the centres of the inscribed and escribed ⊙s of a plane △, then O is the mean centre of the points O′, O″, O‴ for the system of multiples $(s-a)$, $(s-b$, $(s-c)$.

17. In the same case, O′ is the mean centre, for the system s, $s-b$, $s-c$, of the points O′, O″, O‴, and corresponding properties hold for the points O,″ O.‴″

18. If r be the radius of the ⊙ inscribed in a △, and ρ_1, ρ_2 the radii of two ⊙s touching the circumscribed ⊙, and also touching each other at the centre of the inscribed ⊙; then

$$\frac{2}{r} = \frac{}{\rho_1} + \frac{1}{\rho_2}.$$

20. If r, r_1, r_2, r_3 be the radii of the inscribed and escribed ⊙s of a plane △, and R the radius of the circumscribed ⊙; then

$$r_1 + r_2 + r_3 - r = 4\text{R}.$$

21. In the same case, $\dfrac{1}{r} = \dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3}$.

22. In a given ⊙ inscribe a △, so that two of its sides may pass through given points, and that the third side may be a maximum.

23. What theorem analogous to 18 holds for circumscribed $\odot$s?

24. Draw from the vertical $\angle$ of an obtuse-angled $\triangle$ a line to a point in the base whose square will be = rectangle contained by segments of base.

25. If the line AD, bisecting the vertical $\angle$ A of the $\triangle$ ABC, meets the base BC in D, and the circumscribed $\odot$ in E, then the line CE is a tangent to the $\odot$ described about $\triangle$ ABC.

26. The sum of the squares of the $\perp$s from the angular points of a regular polygon inscribed in a $\odot$ upon any diameter of the $\odot$ = half n times the square of the radius.

27. Given the base and vertical $\angle$ of a $\triangle$, find the locus of the centre of the $\odot$ which passes through the centres of the three escribed $\odot$s.

28. If a $\odot$ touch the arcs AC, BC, and the line AB in the construction of Euclid (I. i.), prove its radius = $\frac{3}{8}$ AB.

29. Given the base and the vertical $\angle$ of a $\triangle$, find the locus of the centre of its "Nine-point Circle."

30. If from any point in the circumference of a $\odot$ $\perp$s be let fall on the sides of a circumscribed polygon, the sum of their squares = $\frac{3}{2}$ n times the square of the radius.

31. The internal and external bisectors of the $\angle$s of the $\triangle$, formed by joining the middle points of the sides of another $\triangle$, are the six radical axes of the inscribed and escribed $\odot$s of the latter.

32. The $\odot$ described about a $\triangle$ touches the sixteen $\odot$s inscribed and escribed to the four $\triangle$s formed by joining the centres of the inscribed and escribed $\odot$s of the original $\triangle$.

33. If O, O′ have the same meaning as in question 16, then

$$AO \cdot AO' = AB \cdot AC.$$

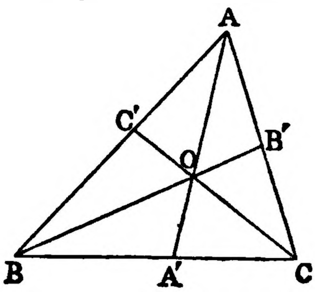
34. Given the base and the vertical $\angle$ of a $\triangle$, find the locus of the centre of a $\odot$ passing through the centre of the inscribed $\odot$ and the centres of any two escribed $\odot$s.

BOOK SIXTH.

SECTION I.

Additional Propositions.

Prop. 1.—*If two $\triangle$s have a common base, but different vertices, they are to one another as the segments into which the line joining the vertices is divided by the common base or base produced.*



Let the two $\triangle$s be AOB, AOC, having the base AO common; let AO cut the line BC, joining the vertices in A′; then

$$\triangle AOB : \triangle AOC :: BA' : A'C.$$

Dem.—$\triangle ABA' : \triangle ACA' :: BA' : A'C$;

$$\triangle OBA' : \triangle OCA' :: BA' : A'C;$$

$$\therefore \triangle ABA' - \triangle OBA' : \triangle ACA' - \triangle OCA' :: BA' : A'C;$$

or $$\triangle AOB : \triangle AOC :: BA' : A'C.$$

Prop. 2.—*If three concurrent lines* AO, BO, CO, *drawn from the angular points of a $\triangle$, meet the opposite sides in the points* A′, B′, C′, *the product of the three ratios*

$$\frac{BA'}{A'C}, \ \frac{CB'}{B'A}, \ \frac{AC'}{C'B} \ \textit{is unity}.$$

Dem.—From the last Proposition, we have

$$\frac{BA'}{A'C} = \frac{\triangle AOB}{\triangle AOC};$$

$$\frac{CB'}{B'A} = \frac{\triangle BOC}{\triangle BOA};$$

$$\frac{AC'}{C'B} = \frac{\triangle COA}{\triangle BOC}.$$

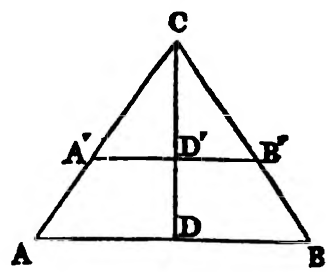
Hence, multiplying out, we get the product = unity.

Cor.—This may be written

$$AB'.\,BC'.\,CA' = A'B\,.\,B'C\,.\,C'A.$$

The symmetry of this expression is apparent. Expressed in words, it gives the product of three alternate segments of the sides = to the product of the three remaining segments.

Prop. 3.—*If two parallel lines be intersected by three concurrent transversals, the segments intercepted by the transversals on the parallels are proportional.*



Let the ||s be AB, A'B', and the transversals CA, CD, CB; then

$$AD : DB :: A'D' : D'B'.$$

Dem.—The $\triangle$s ADC, A'D'C are equiangular;

$$\therefore\ AD : DC :: A'D' : D'C.$$

In like manner $\quad DC : DB :: D'C' : D'B'$;

$\therefore$ *ex aequali* $\quad AD : DB :: A'D' : D'B'$.

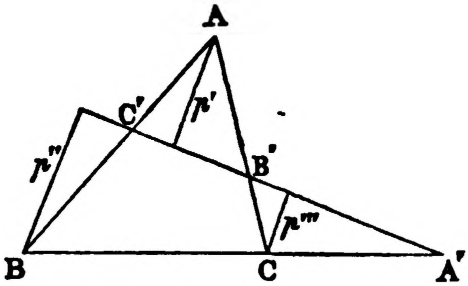
Cor.—If from the points D, D' we draw two $\perp$s DE, D'E' to AC, and two $\perp$s DF, D'F' to BC; then

$$DE : DF :: D'E' : D'F'.$$

Prop. 4.—*If the sides of a triangle* ABC *be cut by any*

transversal, in the points A′, B′, C′; *then the product of the three ratios*

$$\frac{AB'}{B'C'}\ \frac{BC'}{C'A'}\ \frac{CA'}{A'B}$$



is = unity.

Dem. — From the points A, B, C let fall the ⊥s p', p'', p''' on the transversal; then, by similar △s the three ratios are respectively $= \frac{p'}{p'''}, \frac{p''}{p'}, \frac{p'''}{p'}$, and the product of these is evidently = unity. Hence the Proposition is proved.

Observation.—If we introduce the signs, plus and minus, in this Proposition, it is evident that one of the three ratios must be negative. And when the transversal cuts all the sides of the △ externally, all three will be negative. Hence their product will, in all cases, be = to negative unity.

Cor. 1.—If A′, B′, C′ be three points on the sides of a △, either all external, or two internal and one external, such that the product of the three ratios

$$\frac{AB'}{B'C'}\ \frac{BC'}{C'A'}\ \frac{CA'}{A'B'}$$

is = negative unity, then the three points are collinear.

Cor. 2.—The three external bisectors of the ∠s of a △ meet the sides in three points, which are collinear.

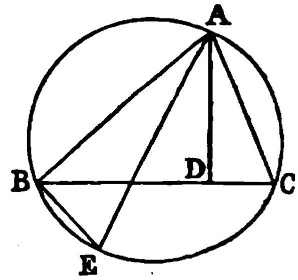
For, let the meeting points be A′, B′, C′, and we have the ratios

$$\frac{BA'}{AC'}, \frac{CB'}{B'A}, \frac{AC'}{CB'}, = \text{to the ratios } \frac{BA}{AC'}, \frac{CB}{BA'}, \frac{AC}{CB'}$$

respectively; and ∴ their product is unity.

Prop. 5.—*In any △, the rectangle contained by two sides is equal to the rectangle contained by the ⊥ on the third side, and the diameter of the circumscribed ⊙:*

Let ABC be the △, AD the ⊥, AE the diameter of the ⊙; then AB . AC = AE . AD.



Dem.—Since AE is the diameter, the ∠ ABE is right, and the ∠ ADC is right; ∴ the ∠ ABE = ∠ ADC; and the ∠ AEB = ∠ ACD (III., 21); ∴ the △s ABE and ADC are equiangular: AB : AE : : AD : AC (iv.). Hence AB . AC = AE . AD.

Cor.—If a, b, c denote the three sides of a △, and R the radius of the circumscribed ⊙, then the area of the $\triangle = \frac{abc}{4R}$.

For, let AD be denoted by p, we have (5)

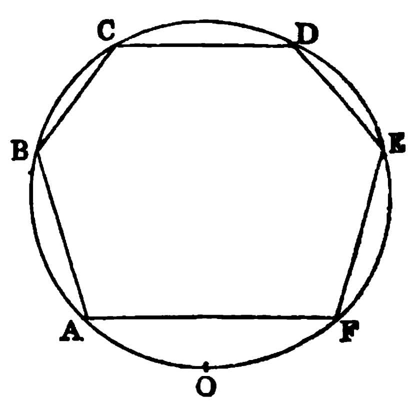
$$2pR = bc;$$

$$\therefore 2apR = abc,$$

$$\frac{ap}{2} = \frac{abc}{4R};$$

that is, area of $\triangle = \frac{abc}{4R}$.

Prop. 6.—*If a figure of any even number of sides be inscribed in a ⊙, the continued product of the ⊥s let fall from any point in the circumference on the odd sides is equal to the continued product of the ⊥s on the even sides.*



We shall prove this Proposition for the case of a hexagon, and then it will be evident that the proof is general.

Let ABCDEF be the hexagon, O the point, and let the ⊥s from O on the lines AB, BC . . . FA, be denoted by α, β, γ, δ, ϵ, ϕ; let △ denote the diameter of the ⊙, and let the lengths of

the six lines OA, OB . . . OF be denoted by l, m, n, p, q, r; then we have $\Delta a = lm$; $\Delta\gamma = np$; $\Delta\epsilon = qr$;

$$\therefore\ \Delta^3 a\gamma\epsilon = lmnpqr.$$

In like manner, $\Delta^3 \beta\delta\phi = lmnpqr$;

$$\therefore\ a\gamma\epsilon = \beta\delta\phi. \qquad \text{(Q.E.D.)}$$

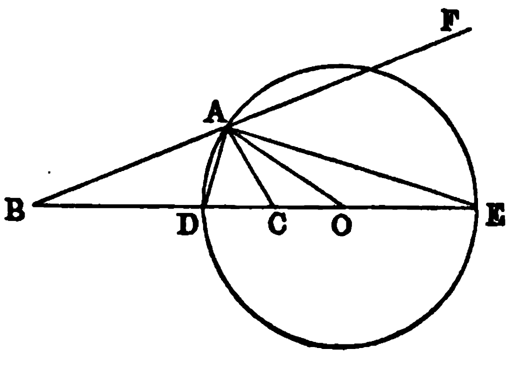
Cor. 1.—The six points A, B, C, D, E, F may be taken in any order of sequence, and the Proposition will hold; or, in other words, if we draw all the diagonals of the hexagon, and take any three lines, such as AC, BD, EF, which terminate in the six points A, B, C, D, E, F, then the continuous product of the ⊥s on them will be = to the continuous product of the ⊥s on any other three lines also terminating in the six points.

Cor. 2.—When the figure inscribed in the ⊙ contains only four sides, this Proposition is the theorem proved (III., 11).

Cor. 3.—If we suppose two of the angular points to become infinitely near; then the line joining these points, if produced, will become a tangent to the ⊙, and we shall in this way have a theorem that will be true for a polygon of an odd number of sides.

Cor. 4.—If ⊥s be let fall from any point in the circumference of a ⊙ on the sides of an inscribed △, their contained product is equal to the contained product of the ⊥s from the same point on the tangents to the ⊙ at the angular points.

Prop. 7.—*Given, in magnitude and position, the base* BC *of a* △ *and the ratio* BA : AC *of the sides, it is required to find the locus of the point* A.



Bisect the internal and the external vertical ∠s by the lines AD, AE. Now, BA : AC : : BD : DC (III.); but the ratio BA : AC is given (Hyp.); ∴ the ratio

BD : DC is given, and BC is given (Hyp.); ∴ the point D is given. In like manner the point E is given. Again, the ∠ DAE is evidently = half the sum of the ∠s BAC, CAF. Hence the ∠ DAE is right, and the ⊙ described on the line DE as diameter will pass through A, and will be the required locus.

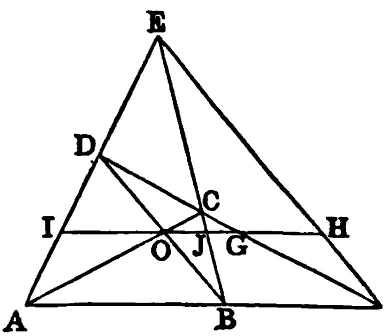
Cor. 1.—The ⊙ described about the △ ABC will cut the ⊙ DAE orthogonally.

For, let O be the centre of the ⊙ DAE. Join AO; then the ∠ DAO = ∠ ADO, that is, ∠ DAC + ∠ CAO = ∠ BAD + ∠ ABO; but ∠ BAD = ∠ DAC; ∴ ∠ CAO = ∠ ABO; ∴ AO touches the ⊙ described about the △ BAC. Hence the ⊙s cut orthogonally.

Cor. 2.—Any ⊙ passing through the points B, C, is cut orthogonally by the ⊙ DAE.

Cor. 3.—If we consider each side of the △ as base in succession, the three ⊙s which are the loci of the vertices have two points common.

Prop. 8.—*If through* O, *the intersection of the diagonals of a quad*[l] ABCD, *a line* OH *be drawn parallel to one of the sides* AB, *meeting the opposite side* CD *in* G, *and the third diagonal in* H, OH *is bisected in* G.



Dem. — Produce HO to meet AD in I, and let it meet BC in J.

Now IJ : JH : : AB : BF, (Prop. 3)

and OJ : JG : : AB : BF;

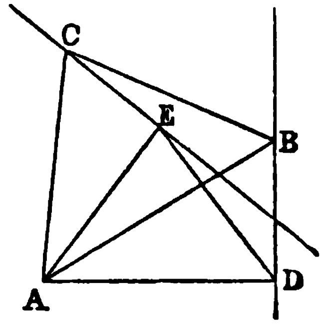
∴ IO : GH : : AB : BF;

but AB : BF : : IO : OG; ∴ OG = GH.

Cor.—GO is a mean proportional between GJ and GI.

Prop. 9.—*If a △ given in species have one angular point fixed, and if a second point moves along a given line, the third will also move along a given line.*

Let ABC be the $\triangle$ which is given in species; let the point A be fixed, the point B move along a given line BD: it is required to find the locus of C.



From A let fall the $\perp$ AD on BD; on AD describe a $\triangle$ ADE equiangular to the $\triangle$ ABC; then the $\triangle$ ADE is given in position; $\therefore$ E is a given point. Join EC. Now, since the $\triangle$s ADE, ABC are equiangular, we have

$$AD : AE :: AB : AC;$$

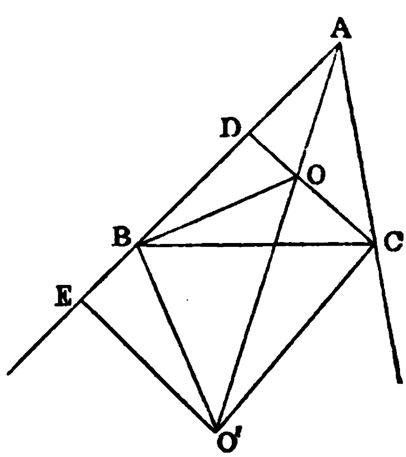
$$\therefore AD : AB :: AE : AC;$$

and the $\angle$ DAB is evidently = $\angle$ EAC. Hence the $\triangle$s DAB, EAC are equiangular; $\therefore$ $\angle$ ADB = $\angle$ AEC. Hence the $\angle$ AEC is right, and the line EC is given in position; $\therefore$ the locus of C is a right line.

Cor.—By an obvious modification of the foregoing Demonstration we can prove the following theorem:—If a $\triangle$ be given in species, and one angular point given in position, then if a second angular point move along a given $\odot$, the locus of the third angular point is a $\odot$.

Prop. 10.—*If* O *be the centre of the inscribed* $\odot$ *of the* $\triangle$ ABC, then $AO^2 : AB \cdot AC :: s - a : s$.

Dem.—Let O′ be the centre of the escribed $\odot$ touching BC externally; let fall the $\perp$s OD, O′E. Join OB, OC, O′B, O′C. Now, the $\angle$s O′BO, O′CO are evidently right $\angle$s; $\therefore$ OBO′C is a quad^l. inscribed in a $\odot$, and $\angle$ BO′O = $\angle$ BCO = $\angle$ ACO; and the $\angle$ BAO′ = $\angle$ OAC. Hence the $\triangle$s O′BA and COA are equiangular; $\therefore$ O′A : BA : AC : AO; $\therefore$ O′A . OA = AB . AC. Hence



$OA^2 : AB . AC :: OA^2 : O'A . OA :: OA : O'A :: AD : AE$; but $AD = s - a$, and $AE = s$;

$$\therefore OA^2 : AB . AC :: (s - a) : s.$$

Cor. 1.— $$\frac{OA^2}{bc} + \frac{OB^2}{ca} + \frac{OC^2}{ab} = 1.$$

For $$\frac{OA^2}{bc} = \frac{s-a}{s}.$$

In like manner, $$\frac{OB^2}{ca} = \frac{s-b}{s},$$

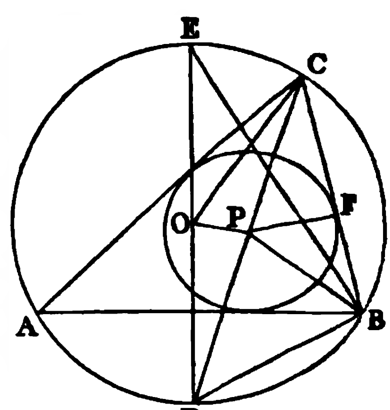
and $$\frac{OC^2}{ab} = \frac{s-c}{s};$$

$\therefore$ by addition, $$\frac{OA^2}{bc} + \frac{OB^2}{ca} + \frac{OC^2}{ab} = 1.$$

Cor. 2.—If O', O'', O''' be the centres of the escribed ⊙s, $$\frac{O'B^2}{ca} + \frac{O'C^2}{ab} - \frac{O'A^2}{bc} = 1, \text{ \&c.}$$

Prop. 11.—*If R, r be the radii of the inscribed and circumscribed ⊙s of a plane triangle, δ the distance between their centres; then*

$$\frac{r}{R+\delta} + \frac{r}{R-\delta} = 1.$$



Dem.—Let O, P be the centres of the ⊙s. Join CP, and let it meet the circumscribed ⊙ in D. Join DO, and produce to meet the circumscribed ⊙ in E. Join EB, OP, PF, PB, BD. Since P is the centre of the inscribed ⊙, CP bisects the ∠ ACB; ∴ the arc AD = the arc DB. Hence the ∠ ABD = ∠ DCB (III., 21); and because PB bisects the ∠ ABC, the ∠ PBA = ∠ PBC; ∴ the ∠ PBD = ∠ PCB + ∠ PBC = ∠ DPB; ∴ DP = DB.

Again, the $\triangle$s DEB, DCF are equiangular; because the $\angle$s DEB and PCF are equal, being in the same segment, and the $\angle$s DBE and PFC are right. Hence DE : DB :: CP : PF (iv.); $\therefore$ DE . PF = DB . PO = DP . PC.

Now, since the $\triangle$ OCD is isosceles, DP . PC = $OC^2 - OP^2$ (ii., 2);

$$\therefore \text{DE} . \text{PF} = OC^2 - OP^2;$$

that is, $$2Rr = R^2 - \delta^2;$$

$$\therefore \frac{r}{R-\delta} + \frac{r}{R+\delta} = 1.$$

Cor. 1.—If r', r'', r''' denote the radii of the escribed $\odot$s, and δ', δ'', δ''' the distances of their centres from the centre of the inscribed $\odot$, we get in like manner

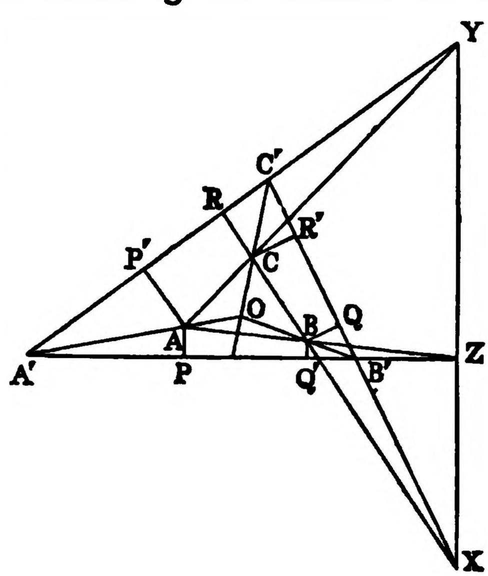
$$\frac{r'}{R-\delta'} + \frac{r'}{R+\delta'} = -1, \text{ \&c.}$$

Cor. 2.—If O′T′, O″T″, O‴T‴ be the tangents from the points O′, O″, O‴ to the circumscribed $\odot$; then

$$2Rr' = O'T'^2, \text{ \&c.}$$

Cor. 3.—If through O we describe a $\odot$, touching the circumscribed $\odot$ and touching the diameter of it, which passes through O, this $\odot$ will be = to the inscribed $\odot$; and similar Propositions hold for $\odot$s passing through the points O′, O″, O‴.

Prop. 12.—*If two triangles be such that the lines joining corresponding vertices are concurrent, then the points of intersection of corresponding sides are collinear.*



Let ABC, A′B′C be the two $\triangle$s, having the lines joining their corresponding vertices meeting in a point O: it is required to prove that the three points X, Y, Z, which are the intersections of corresponding sides, are collinear.

Dem.—From A, B, C let fall three pairs of $\perp$s on the sides of the $\triangle$ A′B′C′; and from O let fall three $\perp$s p', p'', p''' on the sides B′C′, C′A′, A′B′.

Now we have, from *Cor.*, Prop. 3,

$$\frac{AP}{AP'} = \frac{p'''}{p''}, \quad \frac{BQ}{BQ'} = \frac{p'}{p'''}, \quad \frac{CR}{CR'} = \frac{p''}{p'}.$$

Hence the product of the ratios

$$\frac{AP}{AP'}, \quad \frac{BQ}{BQ'}, \quad \frac{CR}{CR'} = \text{unity}. \qquad \text{(iv.)}$$

Again we have, independent of sign,

$$\frac{AZ}{ZB} = \frac{AP}{BQ'}, \quad \frac{BX}{XC} = \frac{BQ}{CR'}, \quad \frac{CY}{AY} = \frac{CR}{AP'}.$$

Hence the product of the three ratios

$$\frac{AZ}{ZB}, \quad \frac{BX}{XC}, \quad \frac{CY}{YA}$$

= the product of the three ratios

$$\frac{AP}{BQ'}, \quad \frac{BQ}{CR'}, \quad \frac{CR}{AP'};$$

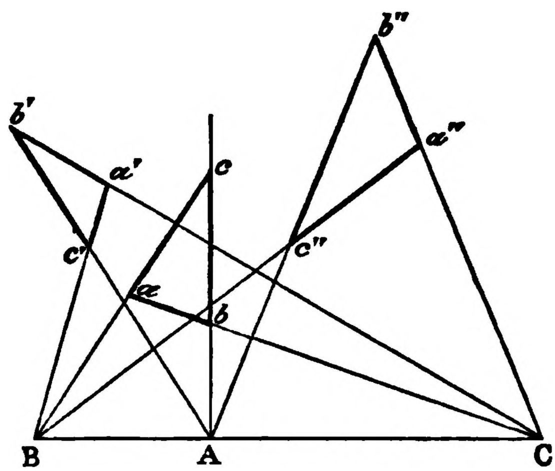
and, therefore, = unity. Hence, by *Cor.*, Prop. 4, the points X, Y, Z are collinear.

Cor.—If two $\triangle$s be such that the points of intersection of corresponding sides are collinear, then the lines joining corresponding vertices are concurrent.

Observation.—Triangles whose corresponding vertices lie on concurrent lines have received different names from geometers.

SALMON and PONCELET call such triangles *homologous*. These writers call the point O the *centre of homology;* and the line XYZ the *axis of homology*. TOWNSEND and CLEBSCH call them triangles in *perspective;* and the point O, and the line XYZ the *centre* and the *axis* of *perspective*.

Prop. 13.—*When three* $\triangle$*s are two by two in perspective, and have the same axis of perspective, their three centres of perspective are collinear.*

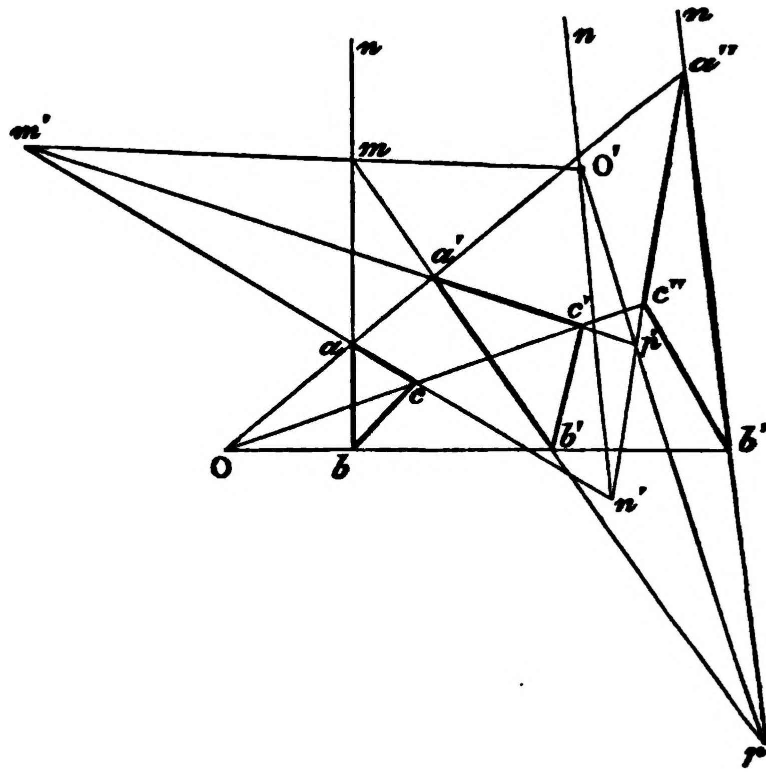


Let abc, $a'b'c'$, $a''b''c''$ be the three $\triangle$s whose corresponding sides are concurrent in the collinear points A, B, C. Now let us consider the two $\triangle$s $aa'a''$, $bb'b''$, formed by joining the corresponding vertices a, a', a'', b, b', b'', and we see that the lines ab, $a'b'$, $a''b''$ joining corresponding vertices are concurrent, their centre of perspective being C. Hence the intersections of their corresponding sides are collinear; but the intersections of the corresponding sides of these $\triangle$s are the centres of perspective of the $\triangle$s abc, $a'b'c'$, $a''b''c''$. Hence the Proposition is proved.

Cor.—The three $\triangle$s $aa'a''$, $bb'b''$, $cc'c''$ have the same axis of perspective; and their centres of perspective are the points A, B, C. Hence the centres of perspective of this triad of $\triangle$s lie on the axis of perspective of the system abc, $a'b'c'$, $a''b''c''$, and conversely.

Prop. 14.—*When three* $\triangle$*s which are two by two in*

perspective have the same centre of homology, their three axes of homology are concurrent.

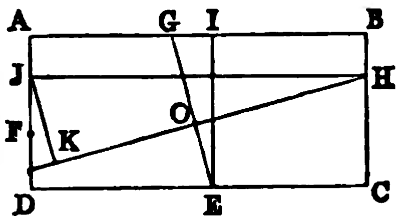


Let *abc*, *a'b'c'*, *a''b''c''* be three $\triangle$s, having the point O as a common centre of perspective. Now, let us consider the two $\triangle$s formed by the two systems of lines *ab*, *a'b'*, *a''b''*; and *ac*, *a'c'*, *a''c''*; these two $\triangle$s are in perspective, the line O*aa'a''* being their axis of perspective. Hence the line joining their corresponding vertices are concurrent, which proves the Proposition.

Cor.—The two systems of $\triangle$s, viz., that formed by the lines *ab*, *a'b'*, *a''b''*; *bc*, *b'c'*, *b''c''*; *ca*, *c'a'*, *c''a''*; and the system *abc*, *a'b'c'*, *a''b''c''*, have corresponding properties—namely, the three axes of perspective of either system meet in the centre of perspective of the other system.

Prop. 15.—We shall conclude this Section with the solution of a few Problems:—

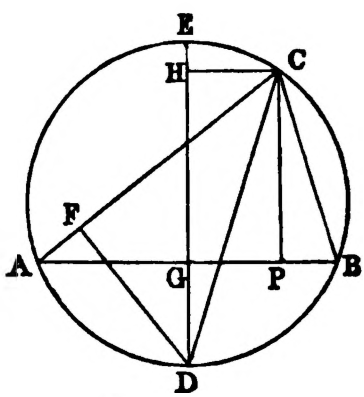
(1). *To describe a rectangle of given area, whose four sides shall pass through four given points.*



Analysis.—Let ABCD be the required rectangle; E, F, G, H the four given points. Through E draw EI || to AD; and through H draw HJ || to AB, and HO ⊥ to EG; and draw JK ⊥ to HO produced.

Now it is evident that the △s EIG, JHK, are equiangular; ∴ the rectangle EI . JH = EG . HK; but EI . JH = area of rectangle, and is given; ∴ the rectangle EG . HK is given, and EG is given; ∴ HK is given. Hence the line KJ is given in position; and since the ∠ FJH is right, the semicircle described on HF will pass through J, and is given in position. Hence the point J, being the intersection of a given line and a given ⊙, is given in position; ∴ the line FJ is given in position.

(2). *Given the base of a △, the perpendicular, and the sum of the sides, to construct it.*



Analysis.—Let ABC be the △, CP the ⊥; and let DE be the diameter of the circumscribed ⊙, which is ⊥ to AB; draw CH || to AB.

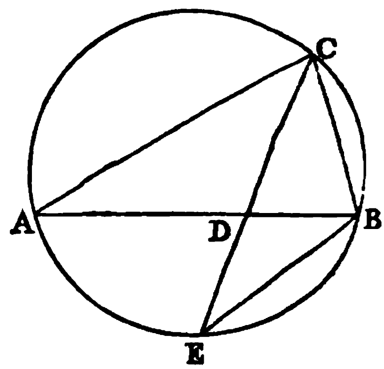
Now the rectangle DH . EG is = to the square of half sum of the sides (iv., 8); ∴ DH . EG is given; and DG . GE = square of GB, and is given. Hence the ratio of DH . GE : DG . GE is given; ∴ the ratio of DH : DG is given. Hence the ratio of GH : DG is given; but GH is = to the ⊥, and is given; hence DG is given; then, if AB be given in position, the point D is given; ∴ the ⊙ ADB is given in position, and GH at a given distance from AP is given in position. Hence the point C is given in position (Q.E.D.)

The method of construction derived from this analysis is evident.

Cor.—If the base, the ⊥, and the difference of the sides be given, a slight modification of the foregoing analysis will give the solution.

(3). *Given the base of a △, the vertical ∠, and the bisector of the vertical ∠, to construct the △.*

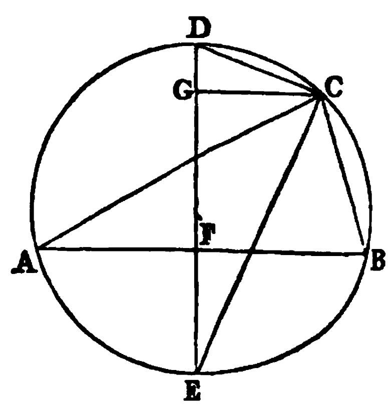
Analysis.—Let ABC be the required △, and let the base AB be given in position; then, since AB is given in position and magnitude, and the ∠ ACB is given in magnitude, the circumscribed ⊙ is given in position. Let CD, the bisector of the vertical ∠, meet the circumscribed ⊙ in E, then E is a given point. Hence EB is given in magnitude.



Now ED . EC = EB^2 (III., 20, *Cor.* 2); ∴ the rectangle ED . EC is given, and CD is given (Hyp.). Hence ED, EC are each given, and the ⊙ described from EC as centre, with EC as radius, is given in position. Hence the point C is given, and the method of construction is evident.

Cor.—From the foregoing we may infer the method of solving the Problem: Given the base, vertical ∠, and external bisector of the vertical ∠.

(4). *Given the base of a △, the difference of the base ∠s, and the difference of the sides, to construct it.*



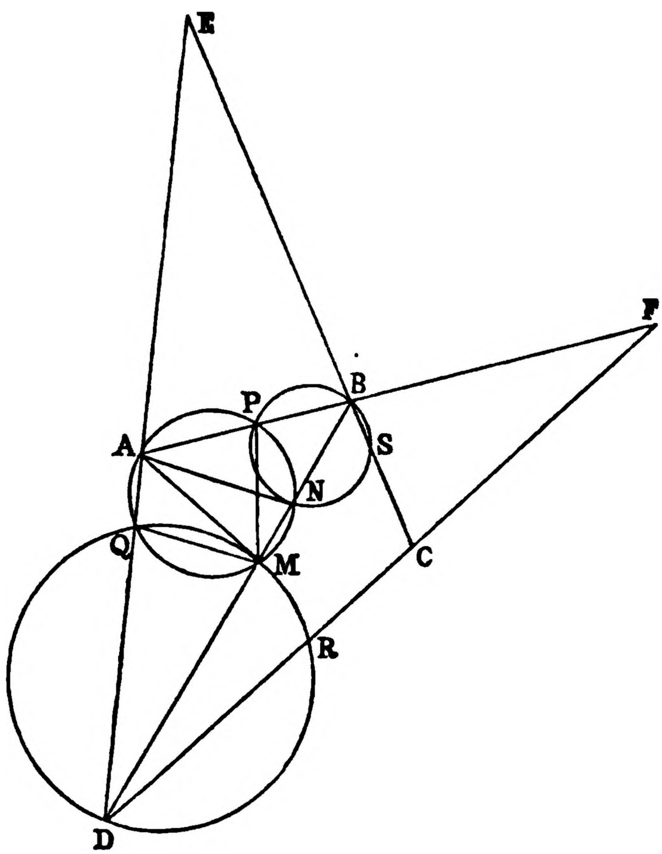
Analysis.—Let ABC be the required △; then the rectangle EF . GD = square of half difference of the sides (IV., 8); ∴ EF . GD is given; and EF . FD = FB^2 is given. Hence the ratio of EF . GD : EF . FD is given. Hence the ratio of FD : GD is given.

Again, the ∠ CED = half the difference of the base ∠s.

and is given; and the $\angle$ DCE is a right $\angle$; $\therefore$ $\triangle$ DCE is given in species, and CGD is equiangular to DCE; $\therefore$ CGD is given in species; $\therefore$ the ratios of GD : DC and of DC : DE are given. Hence the ratio of FD : DE is given; $\therefore$ the ratio of DF : FE is given, and their rectangle is given. Hence DF and FE are each given. Hence the Proposition is solved.

Cor.—In a like manner we may solve the Problem: Given the base, the differnce of the base $\angle$s, and the sum of the sides.

(5). *To construct a quad^l. of given species whose four sides shall pass through four given points.*



Analysis.—Let ABCD be the required quad^l.; P, Q, R, S the four given points. Let E, F be the extremities of the third diagonal. Now, let us consider

the △ ADF; it is evidently given in species, and PQR is an inscribed △ given in species. Hence, if M be the point of intersection of ⊙s described about the △s PAQ, QDR, the △ MAD is given in species.—See Demonstration of (iii., 17).

In like manner, if N be the point of intersection of the ⊙s about the △s QAP, PBS, the △ ABN is given in species. Hence the ratios AM : AD and AN : AB are given; but the ratio of AB to AD is given, because the figure ABCD is given in species. Hence the ratio of AM : AN is given; and M, N are given points; ∴ the locus of A is a ⊙ (7); and where this ⊙ intersects, the ⊙ PAQ is a given point. Hence A is given.

Cor.—A suitable modification of the foregoing, and making use of (III., 16), will enable us to solve the cognate Problem—To describe a quad^1. of given species whose four vertices shall be on four given lines.

(6). *Given the base of a △, the difference of the base ∠s, and the rectangle of the sides, to construct it.*

(7). *Given the base of a △, the vertical ∠, and the ratio of the sum of the sides to the altitude, construct.*

SECTION II.

Centres of Similitude.

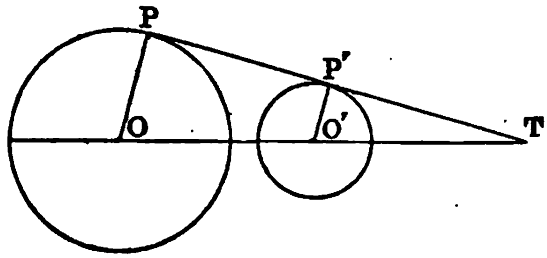
Def.—*If the line joining the centres of two ⊙s be divided internally and externally in the ratio of the radii of the ⊙s, the points of division are called, respectively, the* internal *and the* external *centre of similitude of the two ⊙s.*

From the Definitions it follows that the point of contact of two ⊙s which touch *externally* is an *internal* centre of similitude of the two ⊙s; and the point of contact of two ⊙s, one of which touches another *internally*, is an *external* centre of similitude. Also, since a right line may be regarded as an infinitely

large ⊙, whose centre is at infinity in the direction ⊥ to the line, the centres of similitude of a line and a ⊙ are the two extremities of the diameter of the ⊙ which is ⊥ to the line.

Prop. 1.—*The direct common tangent of two ⊙s passes through their external centre of similitude.*

Dem.—Let O, O′ be the centres of the ⊙s; P, P′ the points of contact of the common tangent; and let PP′ and OO′ produced meet in T; then, by similar △s,



$$OT : O'T :: OP : O'P'.$$

Hence the line OO′ is divided externally in T in the ratio of the radii of the ⊙s; and ∴ T is the external centre of similitude.

Cor. 1.—It may be proved, in like manner, that the transverse common tangent passes through the internal centre of similitude.

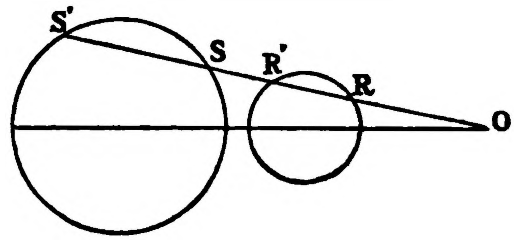
Cor. 2.—The line joining the extremities of parallel radii of two ⊙s passes through their external centre of similitude, if they are turned in the same direction; and through their internal centre, if they are turned in opposite directions.

Cor. 3.—The two radii of one ⊙ drawn to its points of intersection, with any line passing through either centre of similitude, are respectively ∥ to the two radii of the other ⊙ drawn to its intersections with the same line.

Cor. 4.—All lines passing through a centre of similitude of two ⊙s are cut in the same ratio by the ⊙s.

Prop. 2.—*If through a centre of similitude of two ⊙s we draw a secant cutting one of them in the points* R, R′, *and the other in the corresponding points* S, S′; *then*

the rectangles OR . OS′, OR′ . OS *are constant and equal.*



Dem.—Let a, b denote the radii of the ⊙s; then we have (*Cor.* 3, Prop. 2),

$$a : b :: OS : OR;$$

$$\therefore a : b :: OS \,.\, OS' : OR \,.\, OS';$$

but OS . OS′ = square of tangent from O to the ⊙, whose radius is a, and is therefore constant. Hence, since the three first terms of the Proportion are constant, the fourth term is constant.

In like manner, it may be proved that OR′ . OS is a fourth proportional to a, b and OS . OS′; ∴ OR′ . OS is constant.

Prop. 3.—*The six centres of similitude of three ⊙s lie three by three on four lines, called axes of similitude of the ⊙s.*

Dem.—Let the radii of the ⊙s be denoted by a, b, c, their centres by A, B, C; the external centres of similitude by A′, B′, C′, and their internal centres by A″, B″, C″. Now, by Definition,

$$\frac{AC'}{C'B} = -\frac{a}{b};$$

$$\frac{BA'}{A'C} = -\frac{b}{c};$$

$$\frac{CB'}{B'A} = -\frac{c}{a}.$$

Hence the product of the three ratios on the right is negative unity; and ∴ the points A′, B′, C′ are collinear (*Cor.* 1, Prop. 4).

Again, let us consider the system of points A″, B″, C′. We have, as before,

$$\frac{AC'}{C'B} = -\frac{a}{b};$$

$$\frac{BA''}{A''C} = \frac{b}{c};$$

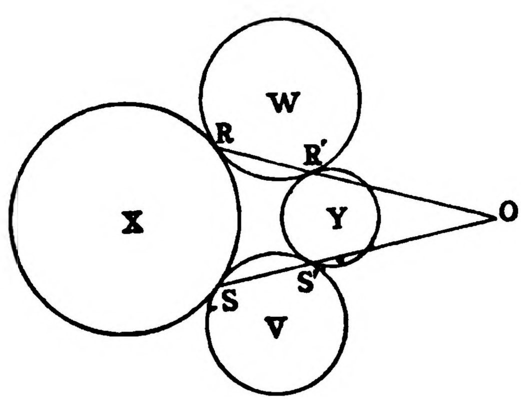
$$\frac{CB''}{B''A} = \frac{c}{a}.$$

Hence the product of the ratios in this case also is negative unity; and ∴ A″, B″, C′ are collinear; and the same holds for A′, B″, C″; A″, B′, C″. Hence the collinearity of centres of similitude will be one external and two internal, or three external centres of similitude.

Cor. 1.—If a variable ⊙ touch two fixed ⊙s, the line joining the points of contact passes through a fixed point, namely—a centre of similitude of the two ⊙s, for the points of contact are centres of similitude.

Cor. 2.—If a variable ⊙ touch two fixed ⊙s, the tangent drawn to it from the centre of similitude through which the chord of contact passes is constant.

Prop. 4.—*If two ⊙s touch two others, the radical axis of either pair passes through a centre of similitude of the other pair.*



Dem.—Let the two ⊙s X, Y touch the two ⊙s W, V; let R, R′ be their points of contact with W, and S, S′ with V. Now, consider the three ⊙s X, W, Y; R, R′ are internal centres of similitude. Hence the line RR′ passes through the external centre of similitude of *X* and *Y*.

In like manner, the line SS′ passes through the same centre of similitude. Hence the point O, where these lines meet, will be the external centre of similitude of *X* and *Y*; and ∴ the rectangle OR . OR′ = OS . OS′ (Prop. 2); ∴ tangent from O to W = tangent from O to V, and the radical axis of W and Y passes through O.

Def.—*The ⊙ on the interval, between the centres of similitude of two ⊙s as diameter, is called their circle of similitude.*

Prop. 5.—*The ⊙ of similitude of two ⊙s is the locus of the vertex of a △ whose base is the interval between the centres of the ⊙s, and the ratio of sides that of their radii.*

Dem.—When the base and the ratio of the sides are given, the locus of the vertex (see Prop. 7, Section I) is the ⊙ whose diameter is the interval between the points in which the base is divided in the given ratio internally and externally; that is, in the present case, the ⊙ of similitude.

Cor. 1.—If from any point in the ⊙ of similitude of two given ⊙s lines be drawu to their centres, these lines are proportional to the radii of the two given ⊙s.

Cor. 2.—If from any point in the ⊙ of similitude of two given ⊙s pairs of tangents be drawn to both ⊙s, the ∠ between one pair is equal to the ∠ between the other pair.

This follows at once from *Cor.* 1.

Cor. 3.—The three ⊙s of similitude of three given ⊙s taken in pairs are coaxal.

For, let P, P′ be the points of intersection of two of the ⊙s of similitude, then it is evident that the lines drawn from either of these points to the centres of the three given ⊙s are proportional to the radii of the given ⊙s. Hence the third ⊙ of similitude must pass through the points P, P′. Hence the ⊙s are coaxal.

Cor. 4.—The centres of the three ⊙s of similitude of three given ⊙s taken in pairs are collinear.

SECTION III.

Theory of Harmonic Section.

Def. — *If a line* AB *be divided internally in the point* C, *and externallly in the point* D, *so that the ratio*

AC : CB = – ratio AD : DB; the points C and D are called harmonic conjugates to the points A, B.

Since the segments AC, CB are measured in the same direction, the ratio AC : CB is positive; and AD, DB being measured in opposite directions, their ratio is negative. This explains why we say AC : CB = – AD : DB. We shall, however, usually omit the sign minus, unless when there is special reason for retaining it.

Cor.—The centres of similitude of two given ⊙s are harmonic conjugates, with respect to their centres.

Prop. 1.—*If* C *and* D *be harmonic conjugates to* A *and* B, *and if* AB *be bisected in* O, *then* OB *is a geometric mean between* OC *and* OD.

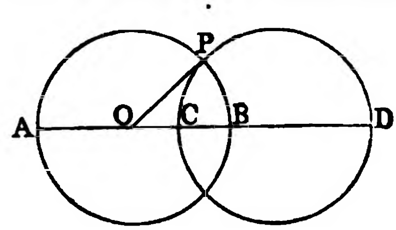
Dem.— AC : CB : : AD : DB;

$$\therefore \frac{AC - CB}{2} : \frac{AC + CB}{2} :: \frac{AD - DB}{2} : \frac{AD + DB}{2};$$

or OC : OB : : OB : OD.

Hence OB is a geometric mean between OC and OD.

Prop. 2.—*If* C *and* D *be harmonic conjugates to* A *and* B, *the ⊙s described on* AB *and* CD *as diameters intersect each other orthogonally.*



Dem.—Let the ⊙s intersect in P, bisect AB in O; join OP; then, by Prop. 2, we have OC . OD = OB² = OP². Hence OP is a tangent to the ⊙ CPD, and ∴ the ⊙s cut orthogonally.

Cor. 1.—Any ⊙ passing through the points C and D will be cut orthogonally by the ⊙ described on AB as diameter.

Cor. 2.—The points C and D are inverse points with respect to the ⊙ described on AB as diameter.

DEF.—*If* C *and* D *be harmonic conjugates to* A *and* B, AB *is called a* harmonic mean *between* AC *and* AD.

Observation.—This coincides with the the algebraic Definition of *harmonic mean.*

For AC, AB, AD being three magnitudes, we have

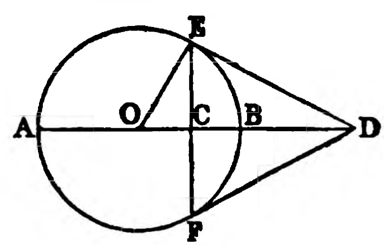
$$AC : CB :: AD : BD;$$
$$\therefore AC : AD :: CB : BD;$$

that is, the 1st : 3rd : : difference between 1st and 2nd : difference between 2nd and 3rd, which is the algebraic Definition.

Cor.—In the same way it can be seen that DC is a harmonic mean between DA and DB.

Prop. 3.—*The Arithmetic mean* : *Geometric mean* : : *Geometric mean* : *Harmonic mean.*

Dem.—Upon AB as diameter describe a ⊙; erect EF at right ∠s to AB through C; draw tangents to the ⊙ at E, F, meeting in D; then, since the △ OED is right-angled at E, and EC is ⊥ to OD, we have OC . OD = OE² = OB². Hence, by Prop. 1, C and D are harmonic conjugates to A and B. Again, from the same △, we have OD : DE : : DE : DC; but OD = ½ (DA + DB) = a. m. between DA and DB; and DE is the g. m. and DC the h. m. between DA and DB.

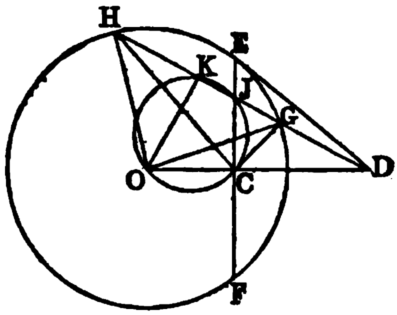


Cor.—The reciprocals of the three magnitudes DA, DO, DB are respectively DB, DC, DA, with respect to DE²; but DA, DO, DB are in arithmetical progression.

Hence the reciprocals of lines in arithmetical progression are in harmonical progression.

Prop. 4.—*Any line cutting a ⊙, and passing through a fixed point, is cut harmonically by the ⊙, the point, and the polar of the point.*

Let D be the point, EF its polar, DGH a line cutting the ⊙ in the points G and H, and the polar of D in the point J; then the points J, D will be harmonic conjugates to H and G.



Dem. — Let O be the centre of the ⊙; from O let fall the ⊥ OK on HD; then, since K and C are right ∠s, OKJC is a quad^1. in a ⊙; ∴ OD . DC = KD . DJ; but OD . DC = DE^2; ∴ KD . DJ = DE^2. Hence KD : DE : : DE : DJ; and since KD, DE are respectively the a. m. and the g. m. between DG and DH, DJ (Prop. 3.) will be the harmonic mean.

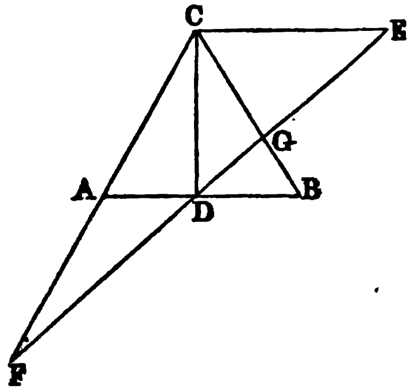
The following is the proof usually given of this Proposition:—Join OH, OG, CH, CG. Now OD . DC = DE^2 = DH . DG; ∴ the quad^1. HOCG is inscribed in a ⊙; ∴ ∠ OCH = ∠ OGH; and ∠ DCG = ∠ OHD; but ∠ OGH = ∠ OHD; ∴ the ∠ OCG = ∠ DCG. Hence the ∠ HCJ = ∠ GCJ; hence CJ and CD are the internal and external bisectors of the vertical ∠ GCH of the △ GCH; ∴ the points J and D are harmonic conjugates to the points H and G. (Q.E.D.)

Cor. 1.—If through a fixed point D any line be drawn cutting the ⊙ in the points G and H, and if DJ be a harmonic mean between DG and DH, the locus of J is the polar of D.

Cor. 2.—In the same case, if DK be the a.m. between DG and DH, the locus of K is a ⊙, namely, the ⊙ described on OD as diameter for the ∠ OKD is right.

Prop. 5.—*If* ABC *be a triangle,* CE *a line through the veterx* ∥ *to the base* AB; *then, any transversal through* D,

the middle of AB, *will meet* CE *in a point, which will be the harmonic conjugate of* D, *with respect to the points in which it meets the sides of the* △.



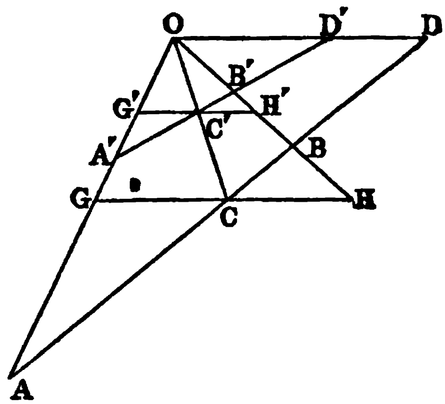
Dem.—From the similar △s FCE, FAD we have EF : FD : : CE : AD; but AD = DB; ∴ EF : FD : : CE : DB.

Again, from the similar △s CEG, DBG, we have CE : DB : : EG : GD;

∴ EF : FD : : EG : GD. (Q.E.D.)

DEFS.—*If we join the points* C, D (see last diagram), *the system of four lines* CA, CD, CB, CE *is called a harmonic pencil; each of the four lines is called a ray; the point* C *is called the vertex of the pencil; the alternate rays* CD, CE *are said to be harmonic conjugates with respect to the rays* CA, CB. *We shall denote such a pencil by the notation* (C . FDGE), *where C is the vertex;* CF, CD, CG, CE *the rays.*

Prop. 6.—*If a line* AB *be cut harmonically in* C *and* D, *and a harmonic pencil* (O . ABCD) *formed by joining the points* A, B, C, D *to any point* O; *then, if through* C *a* ∥ *to* OD, *the ray conjugate to* OC *be drawn, meeting* OA, OB *in* G *and* H, GH *will be bisected in* C.



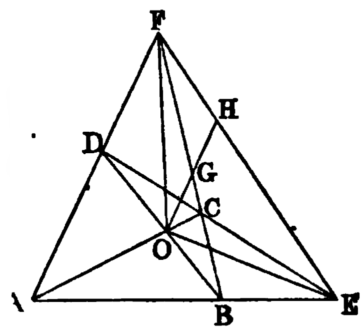
Dem.—

OD : CH : : DB : BC;
and OD : GC : : DA : AC;
but DB : BC : : DA : AC;
∴ OD : CH : : OD : GC. Hence GC = CH.

Cor.—Any transversal A′B′C′D′ cutting a harmonic pencil is cut harmonically.

For, through C′ draw G′H′ ∥ to GH; then, by Prop. 3, Section I., G′C′ : C′H″ : : GC : CH; ∴ G′C′ = C′H′. Hence A′B′C′D′ is cut harmonically.

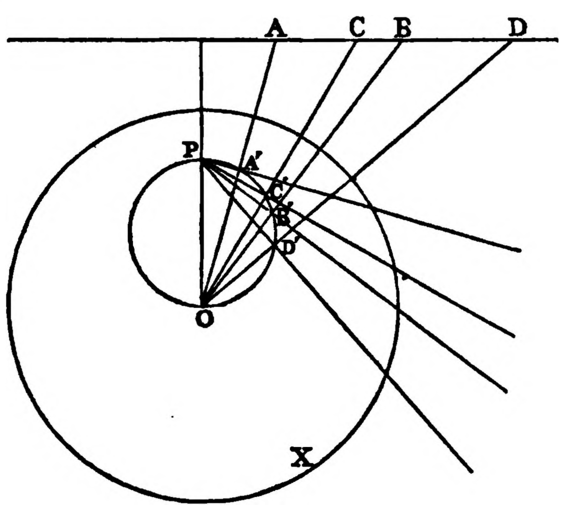
Prop. 7.—*The line joining the intersection of two opposite sides of a quad*[l]*. with the intersection of its diagonals forms, with the third diagonal, a pair of rays, which are harmonic conjugates with these sides.*



Let ABCD be the quad[l]. whose two sides AD, BC meet in F; then the line FO, and the third diagonal FE, form a pair of conjugate rays with FA and FB.

Dem.—Through O draw OH ∥ to AD; meet BC in G, and the third diagonal in H. Then OG = GH (Prop. 8, Section I.). Hence the pencil (F . AOBE) is harmonic. In like manner the pencil (E . AODF) is harmonic.

Prop. 8.—*If four collinear points form a harmonic system, their four polars with respect to any ⊙ form a harmonic pencil.*

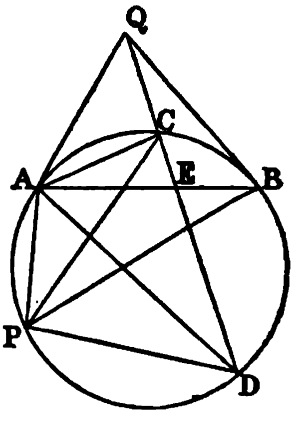


Let A, C, B, D be the four points, P the pole of their line of collinearity with respect to the ⊙ X; let

O be the centre of X. Join OA, OB, OC, OD, and let fall the ⊥s PA′, PB′, PC′, PD′ on these lines; then, by Prop. 25, Section I., Book III., PA′, PB′, PC′, PD′ are the polars of the points A, B, C, D; and since the ∠s at A′, C′, B′, D′ are right, the ⊙ described on OP as diameter will pass through these points; and since the system A, B, C, D is harmonic, the pencil (O . ABCD) is harmonic; but the ∠s between the rays OA, OB, OC, OD are respectively = to the ∠s between the rays PA′, PB′ PC′ PD′ (III., xxi.). Hence the pencil (P . A′B′C′D′) is harmonic.

DEF.—*Four points in a ⊙ which connect with any fifth point in the circumference by four lines, forming a harmonic pencil, are called a harmonic system of points on the ⊙.*

Prop. 9.—*If from any point two tangents be drawn to a ⊙, the points of contact and the points of intersection of any secant from the same point form a harmonic system of points.*



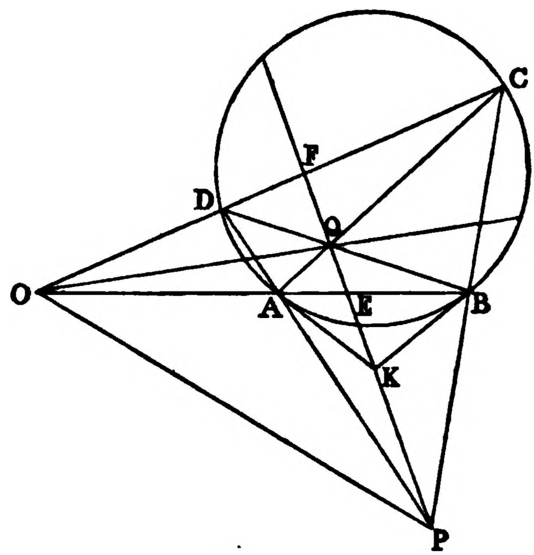
Dem.—Let Q be the point, QA, QB tangents, QCD the secant; take any point P in the circumference of the ⊙, and join PA, PC, PB, PD; then, since AB is the polar of Q, the points E, Q are harmonic conjugates to C and D; ∴ the pencil (A . QCED) is harmonic; but the pencil (P . ACBD) is = to the pencil (A . QCED), for the ∠s between the rays of one = ∠s between the rays of the other; ∴ the pencil (P . ACBD) is harmonic. Hence ACBD form a harmonic system of points.

Cor. 1.—If four points on a ⊙ form a harmonic system, the line joining either pair of conjugates passes through the pole of the line joining the other pair.

Cor. 2.—If the angular points of a quad^{l}. inscribed in a ⊙ form a harmonic system, the rectangle con-

tained by one pair of opposite sides is = to the rectangle contained by the other pair.

Prop. 10.—*If through any point* O *two lines be drawn cutting a* ⊙ *in four points, then joining these points both directly and transversely; and if the direct lines meet in* P *and the transverse lines meet in* Q, *the line* PQ *will be the polar of the point* O.



Dem.—Join OP, then the pencil (P . OAEB) is harmonic (Prop. 7); ∴ the points O, E are harmonic conjugates to the points A, B. Hence the polar of O passes through E (Prop. 4). In like manner, the polar of O passes through F; ∴ the line PQ, which passes through the points E and F, is the polar of O. (Q. E. D.).

Cor. 1.—If we join the points O and Q, it may be proved in like manner that OQ is the polar of P.

Cor. 2.—Since PQ is the polar of O, and OQ the polar of P, then (*Cor.* 1, Prop. 16, Section I., Book III.) OP is the polar of Q.

Def.—*Triangles such as* OPQ, *which possess the property that each side is the polar of the opposite angular point with respect to a given* ⊙, *are called self-conjugate*

△s with respect to the ⊙. Again, if we consider the four points A, B, C, D, *they are joined by three pairs of lines, which intersect in the three points* O, P, Q *respectively; then, on account of the harmonic properties of the quad[l].* ABCD *and the △* OPQ, *I propose to call* OPQ *the harmonic △ of the quad[l].*

Prop. 12.—*If a quad[l]. be inscribed in a ⊙, and at its angular points four tangents be drawn, the six points of intersection of these four tangents lie in pairs on the sides of the harmonic △ of the inscribed quad[l].*

Dem.—Let the tangents at A and B meet in K (see fig., last Prop.), then the polar of the point K passes through O. Hence the polar of O passes through K; ∴ the point K lies on PQ. In like manner, the tangents at C and D meet on PQ. Hence the Proposition is proved.

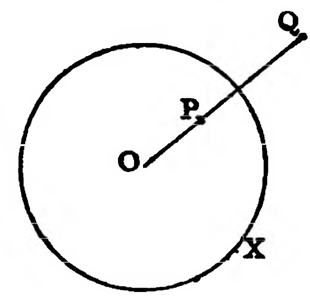
Cor. 1.—Let the tangents at B and C meet in L, at C and D in M, at A and D in N; then the quad[l]. KLMN will have the lines KM (PQ) and LN (OQ) as diagonals; ∴ the point Q is the intersection of its diagonals. Hence we have the following theorem:—*If a quad[l]. be inscribed in a ⊙, and tangents be drawn at its angular points, forming a circumscribed quad[l]., the diagonals of the two quad[ls]. are concurrent, and form a harmonic pencil.*

Cor. 2.—The tangents at the points B and D meet on OP, and so do the tangents at the points A and C. Hence the line OP is the third diagonal of the quad[l]. KLMN; and the extremities of the third diagonal are the poles of the lines BD, AC. Now, since the lines BD, AC are harmonic conjugates to the lines QP, QO, the poles of these four lines form a harmonic system of points. Hence we have the following theorem:—*If tangents be drawn at the angular points of an inscribed quad[l]., forming a circumscribed quad[l]., the third diagonals of these two quad[ls]. are coincident, and the extremities of one are harmonic conjugates to the extremities of the other.*

SECTION IV.

Theory of Inversion.

Def.—*If* X *be a* ⊙, O *its centre*, P *and* Q *two points on any radius, such that the rectangle* OP . OQ = *square of the radius, then* P *and* Q *are called* inverse *points with respect to the* ⊙.

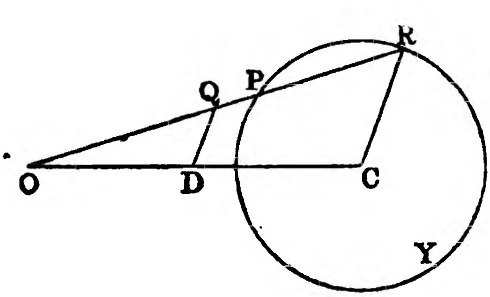


If one of the points, say Q, describe any curve, a ⊙ for instance, the other point P will describe the inverse curve.

We have already given in Book III., Section I., Prop. 20, the inversion of a right line; in Book IV., Section I., Prop. 7, one of its most important applications. This section will give a systematic account of this method of transformation, one of the most important in Geometry.

Prop. 1.—*The inverse of a* ⊙ *is either a line or a* ⊙, *according as the centre of inversion is on the circumference of the* ⊙ *or not on the circumference.*

Dem.—We have proved the first case in Book III.; the second is proved as follows:—Let Y be the ⊙ to be inverted, O the centre of inversion; take any point P in Y; join OP, and make OP . OQ = constant (square of radius of inversion); then Q is the inverse of P: it is required to find the locus of Q. Let OP produced, if necessary, meet the ⊙ Y again at R; then the rectangle OP . OR = square of tangent from O (III. xxxvi.), and ∴ = constant, and OP . OQ is constant (hyp.); ∴ the ratio of OP . OR : OP . OQ is constant: hence the ratio of OR : OQ is constant.



Let C be the centre of Y; join OC, CR, and draw QD ∥ to CR. Now OR : OQ : : CR : QD; ∴ the ratio of CR : QD is constant, and CR is constant; ∴ QD is constant. Again, OR : OQ : : OC : OD; ∴ the ratio of OC : OD is given, and OC is given; ∴ OD is given: hence D is a given point, and DQ is given in magnitude; ∴ the locus of Q is a ⊙, whose centre is the given point D, and whose radius is DQ.

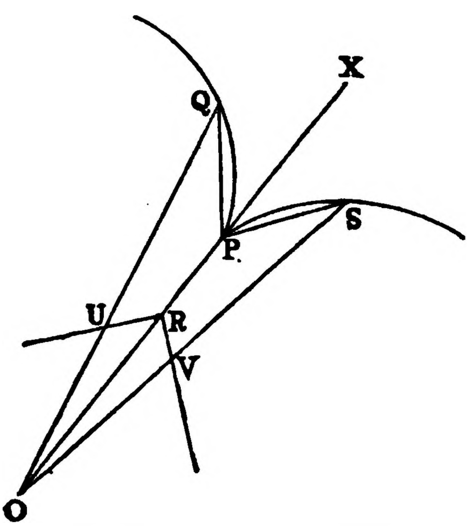
Cor. 1.—The centre of inversion O is the centre of similitude of the original ⊙ Y, and its inverse.

Cor. 2.—The ⊙ Y, its inverse, and the ⊙ of inversion are coaxal. For if the ⊙ Y be cut in any point by the ⊙ of inversion, the ⊙ inverse to Y will pass through that point.

Prop. 2.—***If two ⊙s, or a line and a ⊙, touch each other, their inverses will also touch each other.***

Dem.—If two ⊙s, or a line and a ⊙ touch each other, they have two consecutive points common; hence their inverses will have two consecutive points common, and ∴ they touch each other.

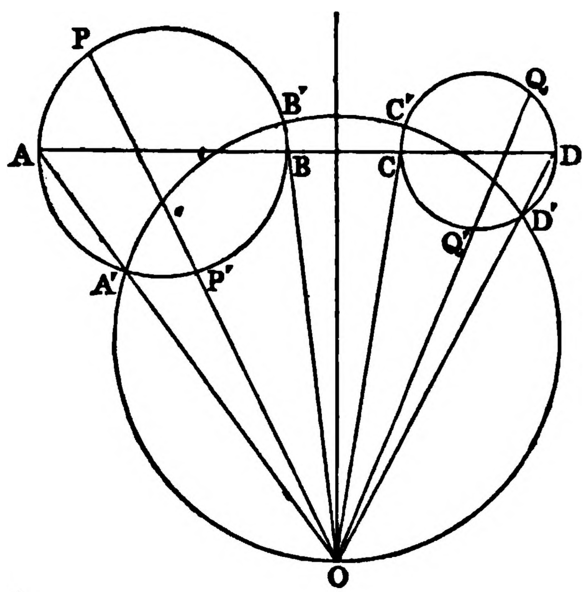
Prop. 3.—***If two ⊙s, or a line and a ⊙, intersect each other, their ∠ of intersection is = to the ∠ of intersection of their inverses.***



Dem.—Let PQ, PS be parts of two ⊙s intersecting in P; let O be the centre of inversion. Join OP; let Q and S be two points on the ⊙s very near P. Join OQ, OS, PQ, PS; and let R, U, V be the inverses of the points P, Q, S. Join UR, VR, and produce OP to X. Now, from the construction, U and V are points on the inverses of the ⊙s PQ, PS. And since the rectangle OP . OR = rectangle OQ . OU, the quad^{l}.

RPQU is inscribed in a ⊙; ∴ the ∠ ORU = ∠ OQP; and when Q is infinitely near P, the ∠ OQP = ∠ QPX; ∴ the ∠ ORU is ultimately = ∠ QPX. In like manner, the ∠ ORV is ultimately = to the ∠ SPX; ∴ the ∠ URV is ultimately = to the ∠ QPS. Now QP, SP are ultimately tangents to their respective ⊙s, and ∴ the ∠ QPS is their ∠ of intersection, and URV is the ∠ of intersection of the inverses of the ⊙s. Hence the Proposition is proved.

Prop. 4.—*Any two ⊙s can be inverted into themselves.*



Dem.—Take any point O in the radical axis of the two ⊙s; and from O draw two lines OPP′, OQQ′, cutting the ⊙s in the points P, P′, Q, Q′; then the rectangle OP . OP′ = rectangle OQ . OQ′ = square of tangent from O to either of the ⊙s, and ∴ = to the square of the radius of the ⊙ whose centre is O, and which cuts both ⊙s orthogonally. Hence the points P′, Q′ are the inverses of the points P and Q with respect to the orthogonal ⊙; and ∴ while the points P, Q move along their respective ⊙s, their inverses, the points P′, Q′ move along other parts of the same ⊙s.

Cor. 1.—The ⊙ of self-inversion of a given ⊙ cuts it orthogonally.

Cor. 2.—Any three ⊙s can be inverted into themselves, their ⊙ of self-inversion being the ⊙ which cuts the three ⊙s orthogonally.

Cor. 3.—If two ⊙s be inverted into themselves, the line joining their centres, namely ABCD, will be inverted into a ⊙ cutting both orthogonally; for the line ABCD cuts the two ⊙s orthogonally.

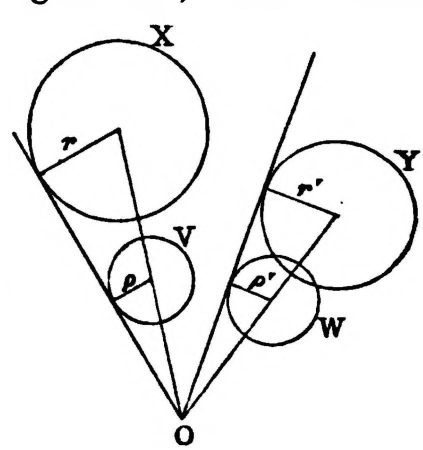
Cor. 4.—Any ⊙ cutting two ⊙s orthogonally may be regarded as the inverse of the line passing through their centres.

Cor. 5.—If ABCD be the line passing through the centres of two ⊙s, and A′B′C′D′ any ⊙ cutting them orthogonally; then the points A′, B′, C′, D′ being respectively the inverses of the points A, B, C, D, the four lines AA′, BB′, CC′, DD′ will be concurrent.

Cor. 6.—Any three ⊙s can be inverted into three ⊙s whose centres are collinear.

Prop. 5.—*Any two ⊙s can be inverted into two equal ⊙s.*

Dem.—Let X, Y be the original ⊙s, r and r' their radii; let V, W be the inverse ⊙s, ρ and ρ' their radii; and let O be the centre of inversion, and T, T′ the tangents from O to X and Y, and R the radius of the circle of inversion. Then, from the Demonstration of Prop. 2, we have



$$r : \rho :: T^2 : R^2;$$
$$r' : \rho' :: T'^2 : R^2.$$

Hence, since $\rho = \rho'$, we have

$$r : r' :: T^2 : T'^2;$$

∴ the ratio of $T^2 : T'^2$ is given; and, consequently, the ratio of $T : T'$ is given. Hence if a point be found,

such that the tangents drawn from it to the two ⊙s X, Y will be in the ratio of the square roots of their radii, the ⊙s X, Y, if inverted from that point, will be equal. It will be seen, in the next Section, that the locus of O is a ⊙ coaxal with X and Y.

Cor. 1.—Any three ⊙s can be inverted into three = ⊙s.

Cor. 2.—Hence can be inferred a method of describing a ⊙ to touch any three ⊙s.

Cor. 3.—If any two ⊙s be the inverses of two others, then any ⊙ touching three out of the four ⊙s will also touch the fourth.

Cor. 4.—If any two points be the inverses of two other points, the four points are concyclic.

Prop. 6.—*If* A *and* B *be any two points,* O *a centre of inversion; and if the inverses of* A, B *be the points* A′, B′, *and* p, p', *the* ⊥*s from* O *on the lines* AB, A′B′; *then* AB : A′B′ : : p : p'.

Dem.—Since O is the centre of inversion, we have

$$OA \,.\, OA' = OB \,.\, OB';$$
$$\therefore OA : OB :: OB' : OA'.$$

And the ∠ O is common to the two △s AOB, A′OB′; ∴ the △s are equiangular. Hence the Proposition is proved.

Prop. 7.—*If* A, B, C . . . L *be any number of collinear points, we have*

$$AB + BC + CD \ldots LA = 0.$$

(Since LA is measured backwards, it is regarded as negative.) Now, let p be the ⊥ from any point O on the line AL; and, dividing by p, we have

$$\frac{AB}{p} + \frac{BC}{p} + \frac{CD}{p} \ldots \frac{LA}{p} = 0.$$

Let the whole be inverted from O; and, denoting the

inverses of the points A, B, C . . . L by A′, B′, C′ . . . L′, we have from the last Article the following general theorem:—*If a polygon* A′B′C′ . . . L′ *of any number of sides be inscribed in a* ⊙, *and if from any point in its circumference* ⊥*s be let fall on the sides of the polygon; then the sum of the quotients obtained by dividing the length of each side by its* ⊥ *is zero.*

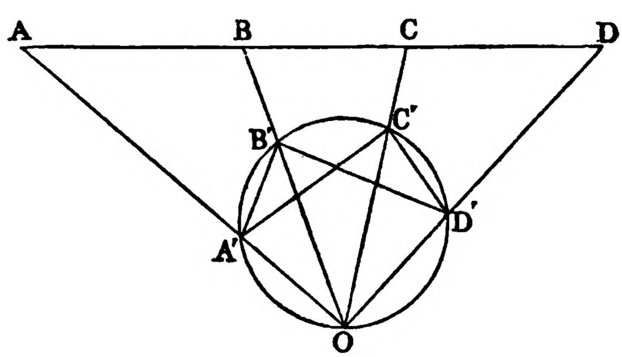
Cor. 1.—Since one of the ⊥s must fall externally on its side of the polygon, while the other ⊥s fall internally, this ⊥ must have a contrary sign to the remainder. Hence the Proposition may be stated thus:—*The length of the side on which the* ⊥ *falls externally, divided by its* ⊥, *is = to the sum of the quotients arising by dividing each of the remaining sides by its perpendicular.*

Cor. 2.—Let there be only three sides, and let the ⊥s be α, β, γ; then, if a, b, c denote the lengths of the sides, &c.,

$$\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 0.$$

Prop. 8.—*If* A, B, C, D *be four collinear points,* A′, B′, C′, D′ *the four points inverse to them; then*

$$\frac{AC \cdot BD}{AB \cdot CD} = \frac{A'C' \cdot B'D'}{A'B' \cdot C'D'}.$$



Dem.—Let O be the centre of inversion, and p the ⊥ from O on the line ABCD; and let the ⊥s from O on the lines A′B′, A′C′, B′D′, C′D′ be denoted by α, β,

γ, δ. Then, by Prop. 7, we have the following equalities:—

$$AC = \frac{A'C' \cdot p}{\beta};$$

$$BD = \frac{B'D' \cdot p}{\gamma};$$

$$AB = \frac{A'B' \cdot p}{\alpha};$$

$$CD = \frac{C'D' \cdot p}{\delta}.$$

Hence multiplying; and remembering that the rectangle $\beta\gamma$ = rectangle $\alpha\delta$ (see Prop. 11, Section I., Book III.), we get

$$\frac{AC \cdot BD}{AB \cdot CD} = \frac{A'C' \cdot B'D'}{A'B' \cdot C'D'}.$$

Cor. 1.—

$$AC \cdot BD : AB \cdot CD : AD \cdot BC$$

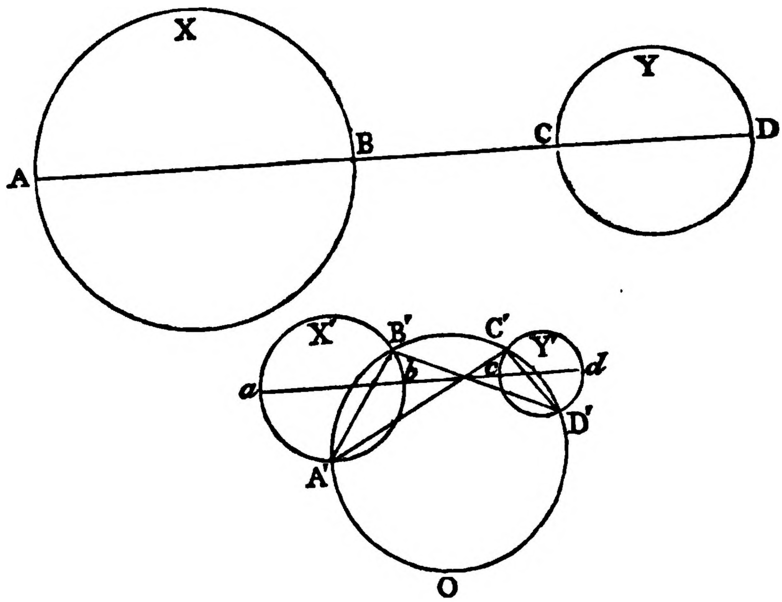
$$:: A'C' \cdot B'D' : A'B' \cdot C'D' : A'D' \cdot B'C'.$$

Cor. 2.—If the points A, B, C, D form a harmonic system, the points A′, B′, C′, D′ form a harmonic system. In other words, the inverse of a harmonic system of points forms a harmonic system.

Cor. 3.—If AB = BC; then the points A′, B′, C′, O form a harmonic system of points.

Prop. 9.—*If two ⊙s be inverted into two others, the square of the common tangent of the first pair, divided by the rectangle contained by their diameters, is = to the square of the common tangent of the second pair, divided by the rectangle contained by their diameters.*

Dem.—Let X, Y be the original ⊙s, X′, Y′ their inverse ⊙s, ABCD the line through the centres of X and Y, and let the inverse of the line ABCD be the ⊙ A′B′C′D′; then, since the line ABCD cuts orthogonally the ⊙s X, Y, its inverse, the ⊙ A′B′C′D′, cuts orthogonally the ⊙s X′, Y′. Let *abcd* be the line through the



centres of the ⊙s X′, Y′; then *abcd* cuts the ⊙s X′, Y′ orthogonally; hence the ⊙ A′B′C′D′ is the inverse of the line *abcd* with respect to a ⊙ of inversion, which inverts the ⊙s X′, Y′ into themselves (see Prop. 4, *Cor.* 3). Hence, by Prop. 9, each of the ratios

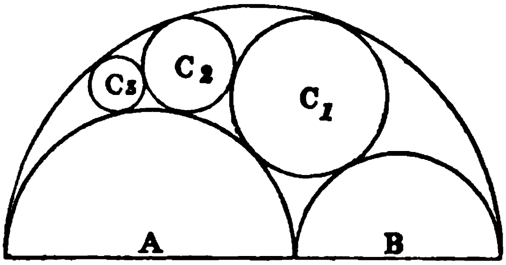
$$\frac{AC \,.\, BD}{AB \,.\, CD}, \quad \frac{ac \,.\, bd}{ab \,.\, cd}$$

is equal to the ratio

$$\frac{A'C' \,.\, B'D'}{A'B' \,.\, C'D'};$$

$$\therefore \quad \frac{AC \,.\, BD}{AB \,.\, CD} = \frac{ac \,.\, bd}{ab \,.\, ca}.$$

The numerators of these fractions are = respectively to the squares of the common tangents of the pairs of ⊙s X, Y; X′, Y′ (see Prop. 8, Section I., Book III). Hence the Proposition is proved.



Cor. 1.—If C_1, C_2, C_3, &c., be a series of ⊙s, touching two parallel lines, and also touching each other; then it is evident, by making the diagram, that the square of the direct common tangent of any two of these ⊙s, such as C_m, C_{m+n}, which are separated by $(n-1)$ circles, is $= n^2$ times the rectangle contained by their diameters. Hence, by inversion and by the theorem of this Article, we have the following theorem:—*If* A *and* B *be any two semicircles in contact with each other, and also in contact with another semicircle, on whose diameter they are described; and if ⊙s* C_1, C_2, C_3 *be described, touching them as in the diagram, the* ⊥ *from the centre of* C_n *on the line* AB = *n times the diameter of* C_n, *where n denotes any of the natural numbers* 1, 2, 3, *&c.*

This theorem will immediately follow by completing the semicircles, and describing another system of circles on the other side = to the system C_1, C_2, C_3, &c., and similarly placed.

Prop. 10.—*If four ⊙s be all touched by the same ⊙; then, denoting by* $\overline{12}$, *the common tangent of the* 1*st and* 2*nd, &c.*,

$$\overline{12}\,.\,\overline{34}+\overline{14}\,.\,\overline{23}=\overline{13}\,.\,\overline{24}.$$

Dem.—Let A, B, C, D be four points taken in order on a right line; then, by Prop. 7, Section I., Book II., we have

$$AB\,.\,CD + BC\,.\,AD = AC\,.\,BD.$$

Now, let four arbitrary ⊙s touch the line at the

points A, B, C, D, and let their diameters be δ, δ', δ'', δ'''; then we have

$$\frac{AB \cdot CD}{\sqrt{\delta\delta'} \cdot \sqrt{\delta''\delta'''}} + \frac{BC \cdot AD}{\sqrt{\delta'\delta''} \cdot \sqrt{\delta\delta'''}} = \frac{AC \cdot BD}{\sqrt{\delta\delta''} \cdot \sqrt{\delta'\delta'''}};$$

and by the last Proposition each of the fractions of this equation remains unaltered by inversion. Hence, if the diameters of the inverse ⊙s be denoted by d, d', d'', d''', and their common tangents by $\overline{12}$, &c., we get

$$\frac{\overline{12} \cdot \overline{34}}{\sqrt{dd'} \cdot \sqrt{d''d'''}} + \frac{\overline{23} \cdot \overline{41}}{\sqrt{d'd''} \cdot \sqrt{d'''d}} = \frac{\overline{13} \cdot \overline{24}}{\sqrt{dd''} \cdot \sqrt{d'd'''}}.$$

Hence $$\overline{12} \cdot \overline{34} + \overline{23} \cdot \overline{14} = \overline{13} \cdot \overline{24}.$$

Cor. 1.—If four arbitrary ⊙s touch a given ⊙ at a harmonic system of points; then

$$\overline{12} \cdot \overline{34} = \overline{23} \cdot \overline{14}.$$

Cor. 2.—The theorem of this Proposition may be written in the form

$$\overline{12} \cdot \overline{34} + \overline{23} \cdot \overline{14} + \overline{31} \cdot \overline{24} = 0;$$

and in this form it proves at once the property of the "Nine-points Circle." For, taking the ⊙s 1, 2, 3, 4 to be the inscribed and escribed ⊙s of the △, and remembering that when ⊙s touch a line on different sides, we are, in the application of the foregoing theorem, to use transverse common tangents. Hence, making use of the results of Prop. 1, Section I., Book IV., we get

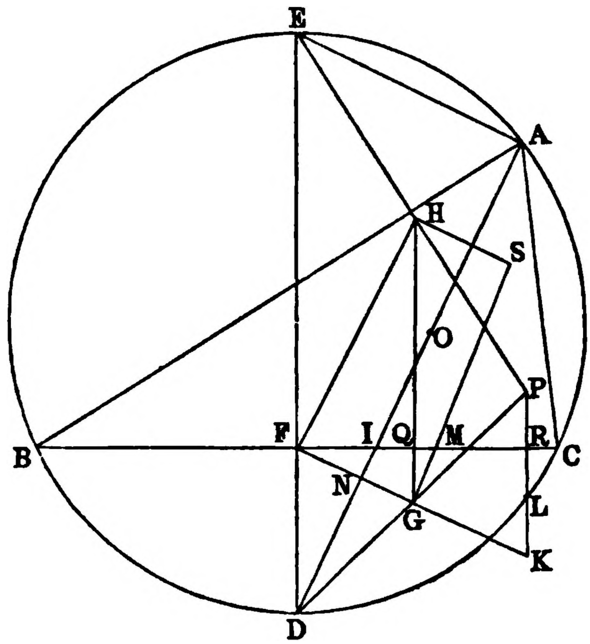
$$\overline{12} \cdot \overline{34} + \overline{23} \cdot \overline{14} + \overline{31} \cdot \overline{24}$$
$$= b^2 - c^2 + c^2 - a^2 + a^2 - b^2 = 0.$$

Hence the ⊙s 1, 2, 3, 4, are all touched by a fifth ⊙.

This theorem is due to Feuerbach. The following simple proof of this now celebrated theorem was pub-

lished by me in the *Quarterly Journal* for February, 1861 :—

"*If* ABC *be a plane* △, *the* ⊙ *passing through the feet of its* ⊥*s touches its inscribed and escribed* ⊙*s.*"



Dem.—Let the inscribed and escribed ⊙s be denoted by O, O′, O″, O‴, the circumscribed ⊙ by X, and the ⊙ through feet of ⊥s by Σ. Now, if P be the intersection of ⊥s, and if the lower segments of ⊥s be produced to meet X, the portions intercepted between P and X are bisected by the sides of ABC (Prop. 13, Section I., Book III.). Hence Σ passes through the points of bisection, and ∴ P is the external centre of similitude of X and Σ.

Let DE be the diameter of X, which bisects BC. Join PD, PE, and bisect them in G and H; then Σ must pass through the points G and H; and since GH is || to DE, GH must be the diameter of Σ; and since Σ passes through F, the middle point of BC (see Prop. 5, Section I., Book IV.), the ∠ GFH is right. Again, if from the point D three ⊥s be let fall on the sides of ABC, their feet are collinear, and the line of colli-

nearity evidently is $\perp$ to AD and it bisects PD (see Prop. 14, Section I., Book III.). Hence FG is the line of collinearity, and FG is $\perp$ to AD. Let M be the point of contact of O with BC; join GM, and let fall the $\perp$ HS. Now, since FM is a tangent to O, if from N we draw another tangent to O, we have $FM^2 = FN^2$ + square of tangent from N (Prop. 21, Section I., Book III.); but $FM = \frac{1}{2}(AB - AC)$. Hence $FM^2 = FR \cdot FI$ (Prop. 8, *Cor.* 5, Section I., Book III.) $= FK \cdot FN$; $\therefore$ square of tangent from $N = FN \cdot NK$. Again, let GT be the tangent from G to O; then GT^2 = square of tangent from $N + GN^2 = FN \cdot NK + GN^2 = GF^2$. Hence the $\odot$ whose centre is G and radius GF will cut the circle O orthogonally; and $\therefore$ that $\odot$ will invert O into itself, and the same $\odot$ will invert BC into Σ; and since BC touches O, their inverses will touch (Prop. 2). Hence Σ touches O, and it is evident that S is the point of contact.

In like manner, if M′ be the point of contact of O′ with BC, and if we join GM′, and let fall the $\perp$ HS′ on GM′, S′ will be the point of contact of Σ with O′.

Cor.—The $\odot$ on FR as diameter cuts the $\odot$s O, O′ orthogonally.

Prop. 11.—Dr. Hart's Extension of Feuerbach's Theorem:—*If the three sides of a plane $\triangle$ be replaced by three $\odot$s, then the $\odot$s touching these, which correspond to the inscribed and escribed $\odot$s of a plane $\triangle$, are all touched by another $\odot$.*

Dem.—Let the direct common tangents be denoted, as in Prop. 11, by $\overline{12}$, &c., and the transverse by $\overline{12'}$, &c., and supposing the signs to correspond to a $\triangle$ whose sides are in order of magnitude a, b, c; then we have, because the side a is touched by the $\odot$ 1 on one side, and by the $\odot$s 2, 3, 4 on the other side,

$$\overline{12'} \cdot \overline{34} + \overline{14'} \cdot \overline{23} = \overline{13'} \cdot \overline{24};$$

$$\overline{12'} \cdot \overline{34} + \overline{24'} \cdot \overline{13} = \overline{23'} \cdot \overline{14};$$

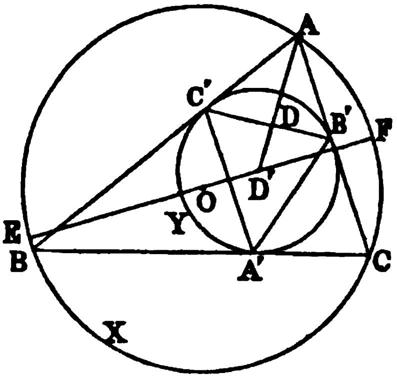
$$\overline{13'} \cdot \overline{24} + \overline{34'} \cdot \overline{12} = \overline{23'} \cdot \overline{14}.$$

Hence $$\overline{14'} \cdot \overline{23} + \overline{34'} \cdot \overline{12} = \overline{24'} \cdot \overline{13};$$

showing that the four $\odot$s are all touched by a $\odot$ having the $\odot$ 4 on one side, and the other three $\odot$s on the other. This proof of Dr. Hart's extension of Feuerbach's theorem was published by me in the *Proceedings of the Royal Irish Academy* in the year 1866.

Prop. 12.—*If two $\odot$s X, Y be so related that a $\triangle$ may be inscribed in X and described about Y, the inverse of X with respect to Y is the "Nine-points Circle" of the $\triangle$ formed by joining the points of contact on Y.*

Dem.—Let ABC be the $\triangle$ inscribed in X and described about Y; and A′B′C′ the $\triangle$ formed by joining the points of contact on Y.



Let O, O′ be the centres of X and Y. Join O′A, intersecting B′C′ in D; then, evidently, D is the inverse of the point A with respect to Y, and D is the middle point of B′C′. In like manner, the inverses of the points B and C are the middle points C′A′ and A′B′; $\therefore$ the inverse of the $\odot$ X, which passes through the points A, B, C with respect to Y, is the $\odot$ which passes through the middle points of B′C′, C′A′, A′B′, that is the "Nine-points Circle" of the $\triangle$ ABC.

Cor. 1.—If two $\odot$s X, Y be so related that a $\triangle$ inscribed in X may be described about Y, the $\odot$ inscribed in the $\triangle$, formed by joining the points on Y, touches a fixed $\odot$, namely, the inverse of X with respect to Y.

Cor. 2.—In the same case, if tangents be drawn to X at the points A, B, C, forming a new $\triangle$ A″B″C″, the $\odot$ described about A″B″C″ touches a fixed $\odot$.

Cor. 3.—Join OO′, and produce to meet the $\odot$ X in the points E and F, and let it meet the inverse of X with respect to Y in the points P and Q; then PQ is the diameter of the "Nine-points Circle" of the $\triangle$

A'B'C', and is ∴ = to the radius of Y. Now, let the radii of X and Y be R, r, and let the distance OO' between their centres be denoted by δ; then we have, because P is the inverse of E, and Q of F,

$$O'P = \frac{r^2}{R+\delta}, \quad O'Q = \frac{r^2}{R-\delta};$$

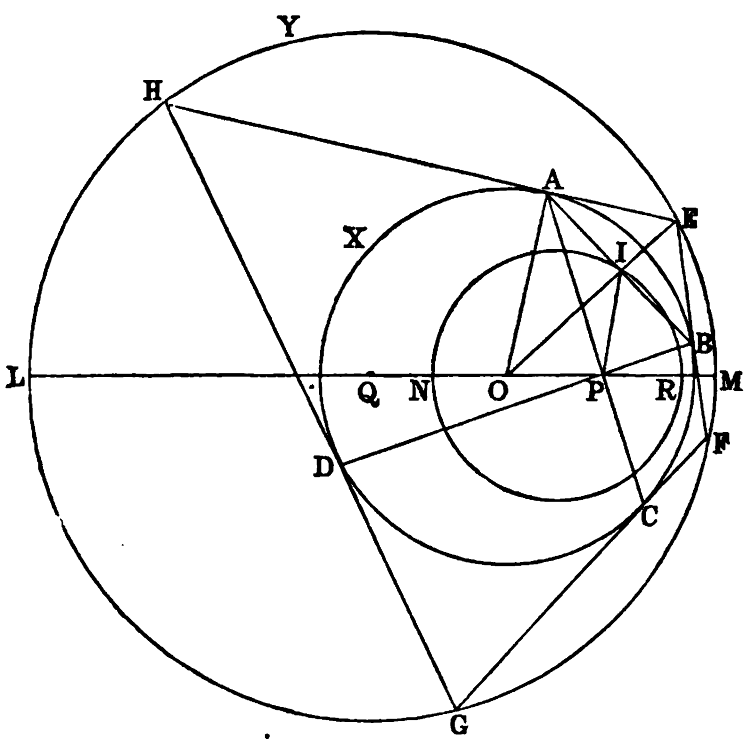
but $$O'P + O'Q = PQ = r;$$

∴ $$\frac{r^2}{R+\delta} + \frac{r^2}{R-\delta} = r.$$

Hence $$\frac{1}{R+\delta} + \frac{1}{R-\delta} = \frac{1}{r};$$

a result already proved by a different method (see Prop. 11, Section I.).

Prop. 13.—*If a variable chord of a ⊙ subtend a right ∠ at a fixed point, the locus of its pole is a ⊙.*



Dem.—Let X be the given ⊙, AB the variable

chord which subtends a right $\angle$ at a fixed point P; AE, BE tangents at A and B, then E is the pole of AB: it is required to find the locus of E. Let O be the centre of X. Join OE, intersecting AB in I; then, denoting the radius of X by r, we have $OI^2 + AI^2 = r^2$; but $AI = IP$, since the $\angle$ APB is right; $\therefore OI^2 + IP^2 = r^2$; $\therefore$ in the $\triangle$ OIP there are given the base OP in magnitude and position, and the sum of the squares of OI, IP in magnitude. Hence the locus of the point I is a $\odot$ (Prop. 2, *Cor.*, Book II.). Let this be the $\odot$ INR. Again, since the $\angle$ OAE is right, and AI is $\perp$ to OE, we have $OI \cdot OE = OA^2 = r^2$. Hence the point E is the inverse of the point I with respect to the $\odot$ X; and since the locus of I is a $\odot$, the locus of E will be a $\odot$ (see Prop. 1).

Prop. 14.—*If two $\odot$s, whose radii are* R, r, *and distance between their centres* δ, *be such that a quad*[l]. *inscribed in one is circumscribed about the other; then*

$$\frac{1}{(R+\delta)^2} + \frac{1}{(R-\delta)^2} = \frac{1}{r^2}.$$

Dem.—Produce AP, BP (see last fig.) to meet the $\odot$ X again in the points C and D; then, since the chords AD, DC, CB subtend right $\angle$s at P, the poles of these chords, viz., the points H, G, F, will be points on the locus of E; then, denoting that locus by Y, we see that the quad[l]. EFGH is inscribed in Y and circumscribed about X. Let Q be the centre of Y; then radius of Y = R, and $OQ = \delta$. Now, since N is a point on the locus of I (see Dem. of last Prop.), $ON^2 + PN^2 = r^2$; but $PN = OR$; $\therefore ON^2 + OR^2 = r^2$. Again, let OQ produced meet Y in the points L and M; then L and M are the inverses of the points N and R with respect to X. Hence

$$ON \cdot OL = r^2; \text{ that is } ON \cdot (R+\delta) = r^2;$$

$$\therefore \qquad ON = \frac{r^2}{R+\delta}.$$

In like manner $\quad OR = \dfrac{r^2}{R - \delta};$

but we have proved $ON^2 + OR^2 = r^2$;

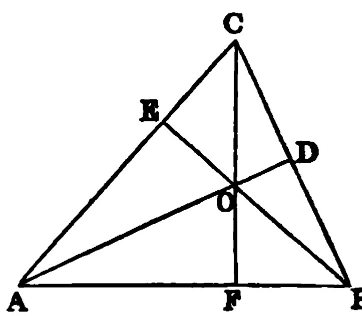
$$\therefore \qquad \frac{r^4}{(R+\delta)^2} + \frac{r^4}{(R-\delta)^2} = r^2;$$

or

$$\frac{1}{(R+\delta)^2} + \frac{1}{(R-\delta)^2} = \frac{1}{r^2}.$$

This Proposition is an important one in the Theory of Elliptic Functions (see Durége, *Theorie der Elliptischen Functionen*, page 185). Our proof is as simple and elementary as could be desired. For another proof, see *Educational Times*, vol. xxxii.

Prop. 15.—*If* ABC *be a plane* △, AD, BE, CF *its* ⊥*s,* O *their point of intersection, then the four* ⊙*s whose centres are* A, B, C, O, *and the squares of whose radii are respectively* = *to the rectangles* AO . AD, BO . BE, CO . CF, OA . OD, *are mutually orthogonal.*



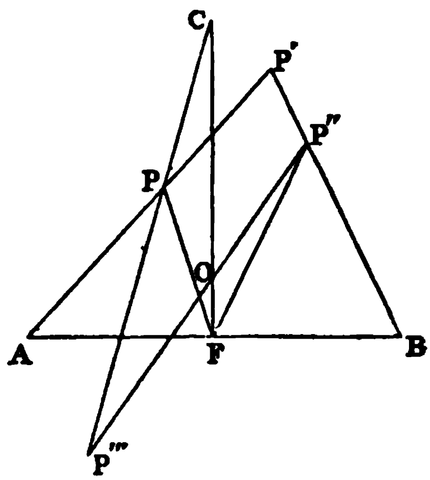
Dem.—AO . AD + BO . BE = AF . AB + BF . BA = AB². Hence the sum of the squares of the radii of the ⊙s whose centres are the points A, B = AB²; ∴ these ⊙s cut orthogonally. Similarly the ⊙s whose centres are O and A cut orthogonally.

Again, let us consider the fourth ⊙, whose centre is the point O, and the square of whose radius is = to the rectangle OA . OD. Now, since OA and OD are measured in opposite directions, they have contrary signs; ∴ the rectangle OA . OD is negative, and the ⊙ has a radius whose square is negative; hence it is imaginary; but, notwithstanding this, it fulfils the condition of intersecting the other ⊙s orthogonally. For AO . AD + OA . OD = AO . AD − AO . OD = AO²; that is, the

sum of the squares of the radii of the ⊙s whose centres are at the points AO = AO^2. Hence these ⊙s cut orthogonally.

Observation.—In this Demonstration we have made the △ acute-angled, and the imaginary ⊙ is the one whose centre is at the intersection of the ⊥s, and the three others are real; but if the △ had an obtuse angle, the imaginary ⊙ would be the one whose centre is at the obtuse angle.

Prop. 16.—*If four ⊙s be mutually orthogonal, and if any figure be inverted with respect to each of the four circles in succession, the fourth inversion will coincide with the original figure.*



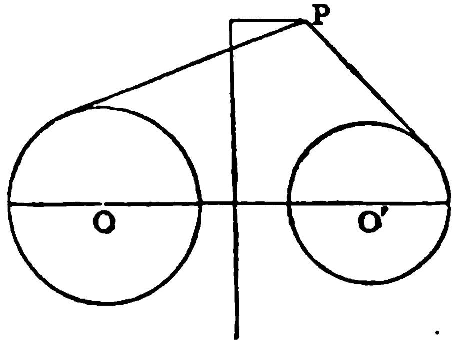
Dem.—It will plainly be sufficient to prove this Proposition for a single point, for the general Proposition will then follow. Let the centres of the four ⊙s be the angular points of A, B, C of a △, and O the intersection of its ⊥s: the squares of the radii will be AB . AF, BA . BF − CO . OF, CF . OF. Now let P be the point we operate on, and let P′ be its inverse with respect to the ⊙ A, and P″ the inverse of P′ with respect to the ⊙ B. Join P″O and CP meeting in P‴. Now, since P′ is the inverse of P with respect to the ⊙ A, the square of whose radius is AB . AF, we have AB . AF = AP . AP′; ∴ the △ AFP is equiangular to the △ AP′B, ∴ ∠AFP = ∠ AP′B: in like manner ∠ BFP″ = ∠ AP′B, ∴ the △s AFP, BP″F are equiangular, ∴ rectangle AF . FB = PF . FP″. Again, because O is the intersection of the ⊥s of the △ ABC, AF . FB = CF . OF. Hence CF . OF = PF . FP″, and the ∠s CFP and OFP″ are equal, since the ∠s AFP and BFP″ are equal; ∴ the △s P″FO and CFP are equiangular, and the ∠s OP″F and PCF are equal; hence the four points C, P″, F, P‴ are concyclic;

$\therefore$ rectangle OP″ . OP‴ = rectangle OC . OF ; the point P‴ is the inverse of P″ with respect to the ⊙ whose centre is O, and the square of whose radius is the negative quantity OC . OD. Again, the ∠ OFP = ∠ P″FO = ∠ OP‴P, $\therefore$ the four points O, F, P‴, P are concyclic; $\therefore$ CP . CP‴ = CO . CF, and the point P is the inverse of P‴ with respect to the ⊙ whose centre is C, and the square of whose radius is the rectangle CF . OF. Hence the Proposition is proved.

SECTION V.

Coaxal Circles.

In Book III., Section I., Prop. 24, we have proved the following theorem:—
"*If from any point* P *tangents be drawn to two* ⊙*s, the difference of their squares is = twice the rectangle contained by the* ⊥ *let fall from* P *on the radical axis and the distance between their centres.*"



The following special cases of this theorem are deserving of notice:—

(1). Let P be on the circumference of one of the ⊙s, and we have—*If from any point* P *in the circumference of one* ⊙ *a tangent be drawn to another* ⊙, *the square of the tangent is = twice the rectangle contained by the distance between their centres and the* ⊥ *from* P *on the radical axis.*

(2). Let the ⊙ to which the tangent is drawn be one of the limiting points, *then the square of the line drawn from one of the limiting points to any point of a* ⊙ *of a coaxal system varies as the* ⊥ *from that point on the radical axis.*

(3). *If* X, Y, Z *be three coaxal* ⊙*s, the tangents drawn from any point of* Z *to* X *and* Y *are in a given ratio.*

(4). *If tangents drawn from a variable point* P *to two given* ⊙*s* X *and* Y *have a given ratio, the locus of* P *is a* ⊙ *coaxal with* X *and* Y.

(5). *The* ⊙ *of similitude of two given* ⊙*s is coaxal with the two* ⊙*s.*

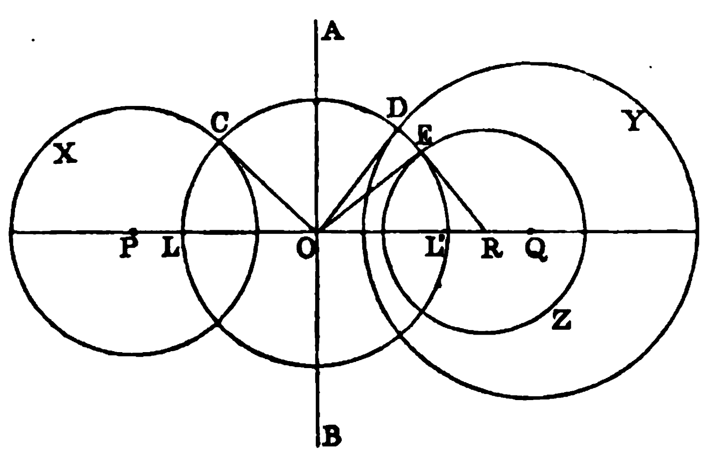
(6). *If* A *and* B *be the points of contact, upon two* ⊙*s* X *and* Y, *of tangents drawn from any point of their* ⊙ *of similitude, then the tangent from* A *to* Y *is = to the tangent from* B *to* X.

Prop. 2.—*Two circles being given, it is required to describe a system of circles coaxal with them.*

Con.—If the ⊙s have real points of intersection, the problem is solved by describing ⊙s through these points and any third point taken arbitrarily.

If the given ⊙s have not real points of intersection, we proceed as follows:—

Let X and Y be the given ⊙s, P and Q their centres: draw AB, the radical axis of X and Y, intersecting PQ in O: from O draw two tangents OC, OD



to X and Y; then OC = OD, and the ⊙ described with O as centre and OD as radius will cut the two ⊙s X and Y orthogonally. Now take any point E in this orthogonal ⊙, and draw the tangent ER meeting the

line PQ in R: from R as centre, and RE as radius, describe a ⊙ Z, then Z will be coaxal with X and Y. For the line ER being a tangent to the ⊙ CDE, the ∠ OER is right, ∴ OE is a tangent to Z; and since OD = OE, the tangents from O to the ⊙s Y and Z are equal: hence OA is the radical axis of Y and Z; ∴ the three ⊙s X, Y, Z are coaxal. In like manner, we can get another ⊙ coaxal with X and Y by taking any other point in the ⊙ CDE, and drawing a tangent and repeating the same construction as with the ⊙ Z. In this way we evidently get two infinite systems of ⊙s coaxal with X and Y, namely, one system at each side of the radical axis. The smallest ⊙ of each system is a point, namely, the point at each side of the radical axis in which the line joining the centres of X and Y cuts the ⊙ CDE. These are the limiting points, and in this view we see that each limiting point is to be regarded as an infinitely small ⊙. The two infinite systems of ⊙s are to be regarded as one coaxal system, the ⊙s of which range from infinitely large to infinitely small—the radical axis being the infinitely large ⊙, and the limiting points the infinitely small.

Cor. 1.—No ⊙ of a system with real limiting points can have its centre between the limiting points.

Cor. 2.—The centres of the ⊙s of a coaxal system are collinear.

Cor. 3.—The ⊙ described on the distance between the limiting points as diameter cuts all the ⊙s of the system orthogonally.

Cor. 4.—Every ⊙ passing through the limiting points cuts all the ⊙s of the system orthogonally.

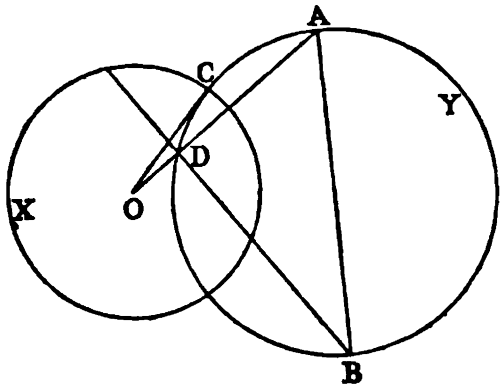
Cor. 5.—The limiting points are inverse points with respect to each ⊙ of the system.

Cor. 6.—The polar of either limiting point, with respect to every ⊙ of the system, is ⊥ to the line of collinearity of centres, and passes through the other.

Prop. 3.—*If two ⊙s* X *and* Y *cut orthogonally, the polar with respect to* X *of any point* A *in* Y *passes through* B, *the point diametrically opposite to* A.

This is Prop. 26, Book III., Section I. The following are important deductions:—

Cor. 1.—The ⊙ described on the line from a point A to any point B in its polar, with respect to a given ⊙, cuts that ⊙ orthogonally.



Cor. 2.—The intersection of the ⊥s of the △ formed by a pair of conjugate points A, B, with respect to a given ⊙ and its centre O, is the pole of the line AB.

Cor. 3.—The polars of any point A with respect to a coaxal system are concurrent. For, through A and through the limiting points describe a ⊙: this (*Cor.* 4, Prop. 2) will cut all the ⊙s orthogonally, and the polars of A with respect to all the ⊙s of the system will pass through the point diametrically opposite to A on this orthogonal ⊙; hence they are concurrent.

Cor. 4.—If the polars of a variable point with respect to three given ⊙s be concurrent, the locus of the point is the ⊙ which cuts the three given ⊙s orthogonally.

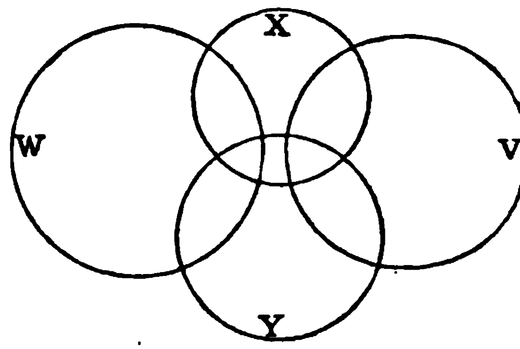
Prop. 4.—*If* X_1, X_2, X_3, &c., *be a system of coaxal* ⊙*s, and if* Y *be any other* ⊙, *then the radical axes of the pairs of* ⊙*s* X_1, Y; X_2, Y; X_3, Y, &c., *are concurrent.*

Dem.—The two first meet on the radical axis of X_1, X_2; the second and third on the radical axis of X_2, X_3; but this, by hypothesis, is the radical axis of X_1, X_2; hence the proposition is evident.

Prop. 5.—*If two* ⊙*s cut two other* ⊙*s orthogonally, the radical axis of one pair is the line joining the centres of the other pair.*

Dem.—Let X, Y be one pair cutting W, V, the other pair, orthogonally; then, since X cuts W and V orthogonally, the tangents drawn from the centre of X to W and V are equal; hence the radical axis of W and V passes

through the centre of X. In like manner the radical axis of W and V passes through the centre of Y; ∴ the line joining the centres of the ⊙s X and Y is the radical axis of the ⊙s W and V. In the same way it can be shown that the line joining the centres of W and V is the radical axis of X and Y.



Cor. 1.—If one pair of the ⊙s, such as W and V, do not intersect, the other pair, X, Y, will intersect, because they must pass through the limiting points of W and V.

Cor. 2.—Coaxal ⊙s may be divided into two classes—one system not intersecting each other in real points, but having real limiting points; the other system intersecting in real points, and having imaginary limiting points.

Cor. 3. If a system of ⊙s be cut orthogonally by two ⊙s they are coaxal.

Cor. 4.—If four ⊙s be mutually orthogonal, the six lines joining their centres, two by two, are also their radical axes, taken two by two.

Prop. 6.—*If a system of concentric ⊙s be inverted from any arbitrary point, the inverse ⊙s will form a coaxal system.*

Dem.—Let O be the centre of inversion, and P the common centre of the concentric system. Through P draw any two lines: these lines will cut the concentric system orthogonally, and ∴ their inverses, which will be two ⊙s passing through the point O and through the inverse of P, will cut the inverse of the concentric system orthogonally; hence the inverse of the concentric system will be a coaxal system (Prop. 5, *Cor.* 3).

Cor. 1.—The limiting points will be the centre of inversion, and the inverse of the common centre of the original system.

Cor. 2.—If a variable ⊙ touch two concentric ⊙s, it will cut any other ⊙ concentric with them at a constant angle. Hence, by inversion, if a variable ⊙ touch two ⊙s of a coaxal system, it will cut any other ⊙ of the system at a constant angle.

Cor. 3.—If a variable ⊙ touch two fixed ⊙s, its radius has a constant ratio to the ⊥ from its centre on the radical axis of the two ⊙s, for it cuts the radical axis at a constant angle.

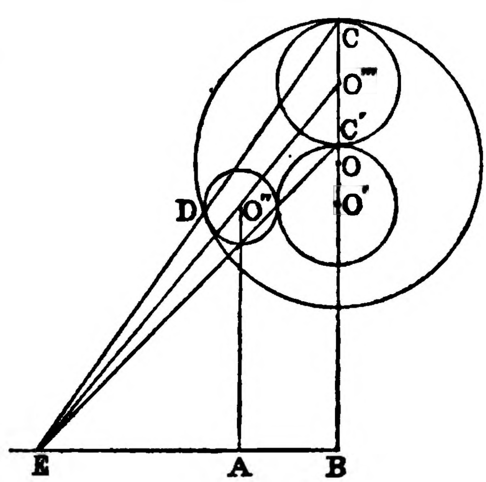
Cor. 4.—The inverse of a system of concurrent lines is a system of coaxal ⊙s intersecting in two real points.

Cor. 5.—If a system of coaxal ⊙s having real limiting points be inverted from either limiting point, they will invert into a concentric system of circles.

Cor. 6.—If a coaxal system of either species be inverted from any arbitrary point, it inverts into another system of the same species.

Prop. 7.—*If a variable ⊙ touch two fixed ⊙s, its radius has a constant ratio to the ⊥ from its centre on the radical axis.*

Dem.—This is *Cor.* 3 of the last Proposition; but it is true universally, and not only as proved there for the case where the ⊙ cuts the radical axis. On account of its importance we give an independent proof here. Let the centres of the fixed ⊙s be O, O′, and that of the variable ⊙ O″. Join OO′, and produce it to meet the fixed ⊙ in the points C, C′: upon CC′ describe a ⊙: let O‴ be its centre: let fall the ⊥s O″A, O‴B on the radical axis: let D be the point of contact of O″ with O; then the lines CD and O‴O″ will meet in the centre of similitude of the ⊙s O″, O‴; but this centre is a



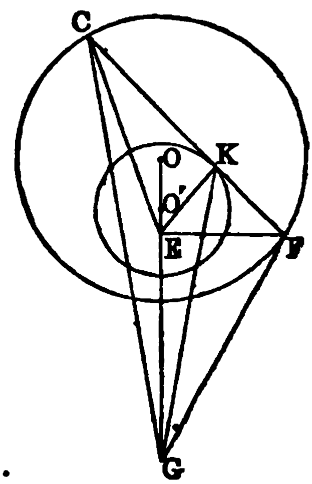
point on the radical axis of the ⊙s O, O′ (see Prop. 5, Section II.). Hence the point E is on the radical axis, and, by similar △s,

$$O''A : O'''B :: O''E : O'''E :: \text{radius of } O'' : \text{radius of } O''',$$
$$\therefore \text{radius of } O'' : O''A :: \text{radius of } O''' : O'''B;$$

but the two last terms of this proportion are constant,

$$\therefore \text{radius of } O'' : O''A \text{ in a constant ratio.}$$

Prop. 8.—*If a chord of one ⊙ be a tangent to another, the ∠ which the chord subtends at either limiting point is bisected by the line drawn from that limiting point to the point of contact.*



Let CF be the chord, K the point of contact, E one of the limiting points: the ∠ CEF is bisected by EG. For since the limiting point E is coaxal with the ⊙s O, O′, we have, by Prop. I. (3),

$$CE : CK :: FE : FK;$$
$$\therefore EC : EF :: KC : KF.$$

Hence the ∠ CEK is bisected (VI. iii).

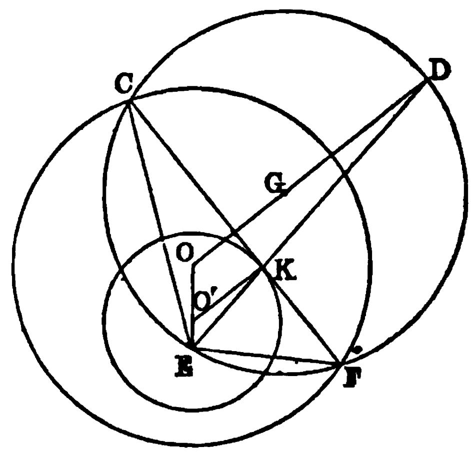
In like manner, if G be the other limiting point, the ∠ CGF is bisected by GK.

Cor. 1.—If the ⊙s were external to each other, and the figure constructed, it would be found that the ∠s bisected would be the supplements of the ∠s CEF, CGF.

Cor. 2.—If a common tangent be drawn to two ⊙s, lines drawn from the points of contact to either limiting point are ⊥ to each other; for they are the internal and external bisectors of an ∠.

Cor. 3.—If three ⊙s be coaxal, a common tangent to two of them will intersect the third in points which are harmonic conjugates to the points of contact; for the pencil from either limiting point will be a harmonic pencil.

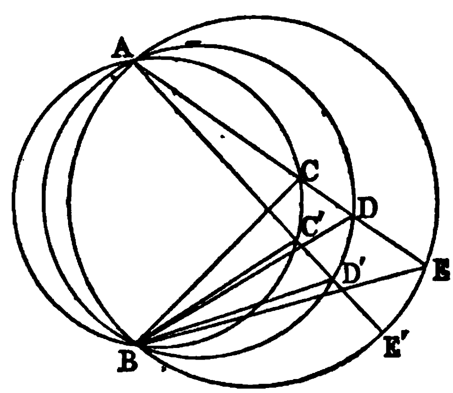
Cor. 4.—If a ⊙ be described about the △ CEF, its envelope will be a ⊙ concentric with the ⊙ whose centre is O; that is, the circle whose chord is CF.



(*When a line or ⊙ moves according to any given law, the curve which it touches in all its positions is called its envelope*).

Produce EK till it meets the circumference in D; then because the ∠ CEF is bisected by ED, the arc CF is bisected in D; hence the line OG, which joins the centres of the ⊙s, passes through D and is ⊥ to CF; ∴ O′K is ∥ to OD; ∴ O′K : OD : : EO′ : EO; hence the ratio of O′K : OD is given, but O′K is given, ∴ OD is given, and the ⊙ whose centre is O and radius OD is given in position, and the ⊙ CEF touches it in D; hence the Proposition is proved.

Prop. 9.—*If a system of coaxal ⊙s have two real points of intersection, all lines through either are divided proportionally by the ⊙s.*



Let A, B be the points of intersection of the coaxal system: through A draw two lines intersecting the ⊙s again in the two systems of points C, D, E; C′, D′, E′; then

CD : DE : : C′D : D′E′.

Dem.—Join the points C, D, E, C′, D′, E′ to B; then the △s BCD, BC′D′ are evidently equiangular, as are

also the $\triangle$s BDE, BD′E′; hence

$$\begin{aligned} &CD : DB :: C'D' : D'B, \\ &DB : DE :: D'B : D'E'; \end{aligned}$$

$\therefore$ *ex aequali*, $CD : DE :: C'D' : D'E'$. Q.E.D.

Cor. 1.—If two lines be divided proportionally, the $\odot$s passing through their point of intersection and through pairs of homologous points are coaxal.

Cor. 2.—If from the point B $\perp$s be drawn to the lines joining homologous points, the feet of these $\perp$s are collinear. For each lies on the line joining the feet of the $\perp$s from B on the lines AC, AC′.

Cor. 3.—The $\odot$s described about the $\triangle$s formed by the lines joining any three pairs of homologous points all pass through B.

Cor. 4.—The intersection of the $\perp$s of all the $\triangle$s formed by the lines joining homologous points are collinear.

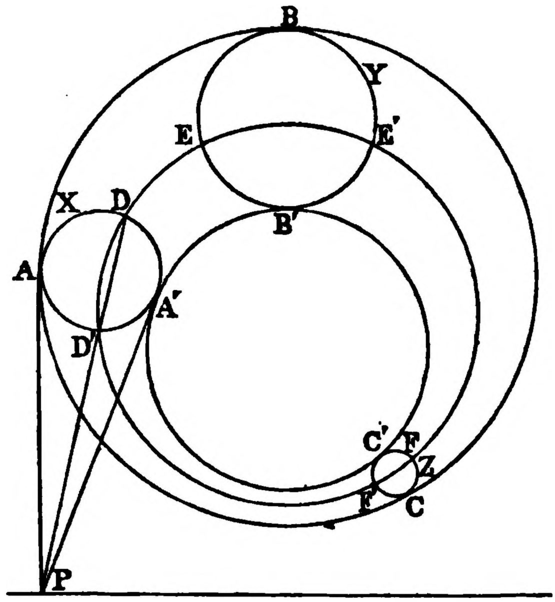
Cor. 5.—Any two lines joining homologous points are divided proportionally by the remaining lines of the system.

Prop. 10.—*To describe a $\odot$ touching three given $\odot$s.*

Analysis.—Let X, Y, Z be the three given $\odot$s, ABC, A′B′C′ two $\odot$s which it is required to describe touching the three given $\odot$s; then, by *Cor.* 2, Prop. 5, Section IV., the $\odot$ DEF which cuts X, Y, Z orthogonally will be the $\odot$ of inversion of ABC, A′B′C′, and the three $\odot$s ABC, DEF, A′B′C′ will be coaxal (*Cor.* 2, Prop. 2, Section IV.).

Now consider the $\odot$ X, and the three $\odot$s ABC, DEF, A′B′C′ the radical axes of X, and these $\odot$s are concurrent (Prop. 4); but two of the radical axes are tangents at A, A′, and the third is the common chord of X and the orthogonal $\odot$ DEF; let P be their point of concurrence. Again, from Prop. 5, Section II., it follows that the axis of similitude of X, Y, Z is the radical axis of the $\odot$s ABC, A′B′C′, but since PA = PA′;

being tangents to X, the point P is on this radical axis. Hence P is the point of intersection of two given lines, namely, the axis of similitude of X, Y, Z, and the chord common to X and the orthogonal ⊙ DEF; ∴ P is a given point; hence A, A′, the points of contact of tangents from P to X, are given. Similarly, the points



B, B′; C, C′ are given points. And we have the following construction, viz.: *Describe the orthogonal ⊙ of* X, Y, Z, *and draw the three chords of intersection of this* ⊙ *and* X, Y, Z *respectively; and from the points where these chords meet the axis of similitude of* X, Y, Z, *draw pairs of tangents to* X, Y, Z; *then the two* ⊙*s through these six points of contact will be tangential to* X, Y, Z.

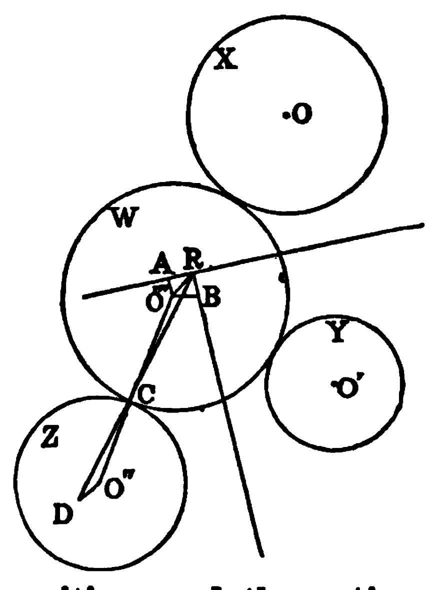
Cor. 1.—Since there are four axes of axes of similitude of X, Y, Z, we shall have eight ⊙s tangential to X, Y, Z.

Cor. 2.—If we suppose one of the ⊙s to reduce to a point, we have the problem: "*To describe a* ⊙ *touching two given* ⊙*s, and passing through a given point.*" And if two of the ⊙s reduce to points, we have the problem: "*To describe a* ⊙ *touching a given* ⊙, *and passing through two given points.*"

The foregoing construction holds for each case, the first of which admits of four solutions, and the second of two.

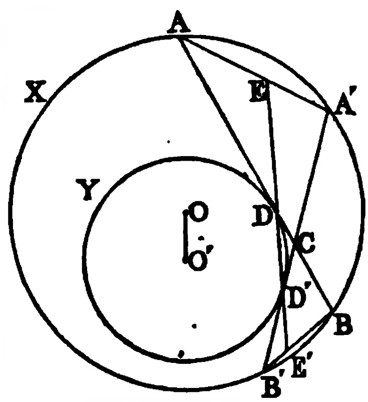
Cor. 3.—Similarly, we may suppose one of the ⊙s to open out into a line, and we have the problem : "*To describe a ⊙ touching a line and two given ⊙s*"; and if two ⊙s open out into lines, the problem: "*To describe a ⊙ touching two given lines and a ⊙.*" The foregoing construction extends to these cases also, and like observations apply to the remaining cases, namely, when one of the ⊙s reduces to a point, and one opens out into a line, &c. Since our construction embraces all cases, except where the three circles become three points or open out into three lines, it would appear to be the most general construction yet given for the solution of this celebrated problem.

Another Method—**Analysis.**—Let O, O′, O″ be the centres of the ⊙s X, Y, Z, and let AR, BR be the radical axis of the pairs of ⊙s XY, YZ, respectively, and let O‴ be the centre of the required ⊙ W: from O‴ let fall the ⊥s O‴A, O‴B; join R to C, the point of contact of W with Z, and produce it to meet O″D drawn ∥ to O‴R. Now, because W touches the ⊙s X, Y, its radius O‴C has a given ratio to O‴A (Prop. 7). Similarly, O‴C has a given ratio O″B, ∴ O‴A has a given ratio to O″B; hence the line O‴R is given in position, and the ratio of O‴R : O‴B is given; ∴ the ratio of O‴R : O‴C is given; hence the ratio of O″D : O″C is given; ∴ D is a given point and R is a given point; ∴ the line RD is given in position; hence C is a given point. Similarly, the other points of contact are given.



Observation.—This method, though arrived at by the theory of coaxal ⊙s, is virtually the same as Newton's 16th Lemma. It is, however, somewhat simpler, as it does not employ conic sections, as is done in the *Principia*. When I discovered it several years ago, I was not aware to what an extent I had been anticipated.

Prop. 11.—*If* X, Y *be two* ⊙*s*, AB, A′B′ *two chords of* X *which are tangents to* Y; *then if the* ⊥*s from* A, A′ *on the radical axis be denoted by* p, π, *and the* ⊥*s from* B, B′ *by* p', π',



$$AA' : BB' :: \sqrt{p} + \sqrt{\pi} : \sqrt{p'} + \sqrt{\pi'}.$$

Dem.—Let O, O′ be the centres of the ⊙s; then, by (1), Prop. 1.,

$$AD = \sqrt{2 . OO' . p}, \qquad A'D' = \sqrt{2 . OO' . \pi};$$

$$\therefore AD + A'D' = \sqrt{2 . OO'}\,\{\sqrt{p} + \sqrt{\pi}\}.$$

But AD + A′D′ is easily seen to be = AC + A′C;

$$\therefore AC + A'C = \sqrt{2 . OO'}\,\{\sqrt{p} + \sqrt{\pi}\}.$$

In like manner,

$$BC + B'C = \sqrt{2 . OO'}\,\{\sqrt{p'} + \sqrt{\pi'}\}.$$

Hence,

$$AC + A'C : BC + B'C :: \sqrt{p} + \sqrt{\pi} : \sqrt{p'} + \sqrt{\pi'}.$$

Now, since the △s AA′C, BB′C are equiangular, we have

$$AC + A'C : BC + B'C :: AA' : BB';$$

$$\therefore AA' : BB' :: \sqrt{p} + \sqrt{\pi} : \sqrt{p'} + \sqrt{\pi'}.$$

This theorem is very important, besides leading to an immediate proof of *Poncelet's Theorem*. If we suppose

the chords AB, A′B′ to be indefinitely near, we can infer from it a remarkable property of the motion of a particle in a vertical circle, and also a method of representing the amplitude of Elliptic Integrals of the First kind by coaxal circles.

Prop. 12.—Poncelet's Theorem. *If a variable polygon of any number of sides be inscribed in a ⊙ of a coaxal system, and if all the sides but one in every position touch fixed ⊙s of the system, that one also in every position touches another fixed ⊙ of the system.*

It will be sufficient to prove this Theorem for the case of a △, because from this simple case it is easy to see that the theorem for a polygon of any number of sides is an immediate consequence.

Let ABC be a △ inscribed in a ⊙ of the system, A′B′C′ another position of the △, and let the sides AB, A′B′ be tangents to one ⊙ of the system, BC, B′C′ tangents to another ⊙; then it is required to prove that CA, C′A′ will be tangents to a third ⊙ of the system.

Dem.—Let the ⊥s from A, B, C on the radical axis be denoted by p, p', p'', and the ⊥s from A′, B′, C′ by π, π', π''; then, by Prop. 11, we have

$$AA' : BB' :: \sqrt{p} + \sqrt{\pi} : \sqrt{p'} + \sqrt{\pi'},$$

and $$BB' : CC' :: \sqrt{p'} + \sqrt{\pi'} : \sqrt{p''} + \sqrt{\pi''};$$

$$\therefore AA' : CC' :: \sqrt{p} + \sqrt{\pi} : \sqrt{p''} + \sqrt{\pi''}.$$

Hence AC, A′C are tangents to another ⊙ of the system.

The foregoing proof of this celebrated theorem was given by me in 1858 in a letter to the Rev. R. Townsend, F.T.C.D. It is virtually the same as Dr. Hart's proof, published in 1857 in the *Quarterly Journal of Mathematics*, of which I was not aware at the time.

Dr. Hart's Proof. This proof depends on the following Lemma (see fig., Prop. 11) :—If a quadrilateral

AA′BB′ be inscribed in a ⊙ X, and if the diagonals AB, A′B′ touch a ⊙ Y of a system coaxal with X, then the sides A, A′ touch another ⊙ of the same system, and the four points of contact D, D′, E, E′ are collinear.

This proposition is evident from the similar △s AED, B′E′D′, and the similar △s EA′D′, E′BD; and the equality of the ratios AE : AD, B′E′ : B′D′, A′E : A′D, BE : BD.

The first part of this theorem also follows at once from Prop. 11.

Now, to prove Poncelet's Theorem:—Let ABC, A′B′C′ be two positions of the variable △, and let, as before, AB, A′B′ be tangents to one ⊙ of the system, BC, B′C′ tangents to another ⊙; then CA, C′A′ shall be tangents to a third ⊙ of the system. For, join AA′, BB′, CC′. Then, since AB, A′B are tangents to a ⊙ of the system, AA , BB′ are, by the lemma, tangents to another ⊙ of the system; and since BC, B′C′ are tangents to a ⊙ of the system, BB′, CC′ are tangents to a ⊙ of the system; ∴ AA, BB′, CC′ are tangents to a ⊙ of the system; and since AA′, CC′ touch a ⊙ of the system by the lemma, AC, A′C′ touch a ⊙ of the system; hence the proposition is proved, and we see that the two proofs are substantially identical.

SECTION VI.

Theory of Anharmonic Section.

Def.—*A system of four collinear points* A, B, C, D *make, as is known, six segments; these may be arranged in three pairs, each containing the four letters—thus,*

AB, CD; BC, AD; CA, BD.

Where the last letter in each couple is D, *and the first segments in the three couples are respectively* AB, BC,

CA, *exactly corresponding to the sides of a* △ ABC, *taken in order. Now, if we take the rectangles formed by these three pairs of segments, the six quotients obtained by dividing each rectangle by the two remaining ones are called the six anharmonic ratios of the four points* A, B, C, D. *Thus these six functions are*

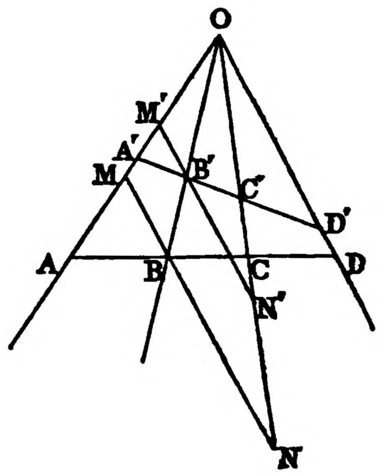
$$\frac{AB \cdot CD}{BC \cdot AD}, \quad \frac{BC \cdot AD}{CA \cdot BD}, \quad \frac{CA \cdot BD}{AB \cdot CD};$$

and their reciprocals

$$\frac{BC \cdot AD}{AB \cdot CD}, \quad \frac{CA \cdot BD}{BC \cdot AD}, \quad \frac{AB \cdot CD}{CA \cdot BD}.$$

It is usual to call any one of these six functions the anharmonic ratio of the four points A, B, C, D.

Prop. 1.—*If* (O . ABCD) *be a pencil of four rays passing through the four points* A, B, C, D; *and if through any of these points* B *we draw a line* || *to a ray passing through any of the other points, and cutting the two remaining rays in the points* M, N, *the six anharmonic ratios of* A, B, C, D *can be expressed in terms of the ratios of the segments* MB, BN, NM.



Dem.—From similar △s,

$$\frac{MB}{OD} = \frac{AB}{AD},$$

and

$$\frac{OD}{BN} = \frac{CD}{BC}.$$

Hence,

$$\frac{MB}{BN} = \frac{AB \cdot CD}{BC \cdot AD};$$

$$\therefore MB : BN :: AB \cdot CD : BC \cdot AD.$$

Componendo—

$$MN : BN :: AB \cdot CD + BC \cdot AD : BC \cdot AD;$$

$$\therefore MN : BN :: AC \cdot BD : BC \cdot AD;$$

$$\therefore MB : BN : NM :: AB \cdot CD : BC \cdot AD : CA \cdot BD.$$

(Q. E. D.)

Prop. 2.—*If a pencil of four rays be cut by two transversals* ABCD, A′B′C′D′, *then* (see last fig.) *any of the anharmonic ratios of the points* A, B, C, D *is equal to the corresponding ratio for the points* A′, B′, C′, D′.

Dem.—Through the points B, B′ draw MN, M′N′ || to OD; then (Section I., Prop. 3) we have

$$MB : BN :: M'B' : B'N';$$

$$\therefore \frac{AB \cdot CD}{BC \cdot AD} = \frac{A'B' \cdot C'D'}{B'C' \cdot A'D'}. \qquad \text{(Q. E. D.)}$$

Cor. 1.—We may suppose the rays of the pencil produced through the vertex, and the transversal to cut any of the rays produced without altering the anharmonic ratio.

Def.—*The anharmonic ratio of the four points on any transversal cutting a pencil being constant, it is called the anharmonic ratio of the pencil.*

Cor. 2.—If two pencils have equal anharmonic ratios and a common vertex; and if three rays of one pencil be the production of three rays of the other, then the fourth ray of one is the production of the fourth ray of the other.

Cor. 3.—If two pencils have a common transversal, they are equal; that is, they have = anharmonic ratios.

Cor. 4.—If A, B, C, D be four points in the circumference of a ⊙, and E and F any other two points also in the circumference, then the pencil (E . ABCD) = (F . ABCD). This is evident, since the pencils have equal angles.

Def. — *The anharmonic ratio of a cyclic pencil* (E . ABCD) *is called the anharmonic ratio of the four cyclic points* A, B, C, D.

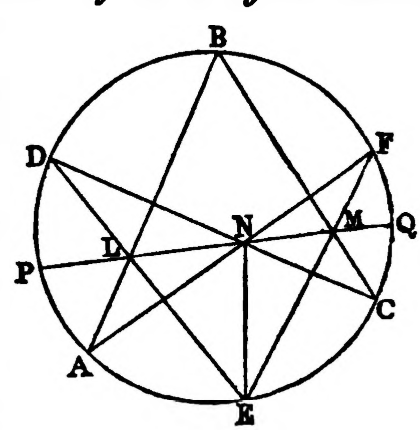
Prop. 3.—*The anharmonic ratio of four concyclic points can be expressed in terms of the chords joining these four points.*

Dem. (see fig., Prop. 9, Section IV.)—The anharmonic ratio of the pencil (O . ABCD) is AC . BD : AB . CD; and this, by Prop. 9, Section IV. = A′C′ . B′D′ : A′B′ . C′D′; but the pencil (O . ABCD) = pencil (O . A′B′C′D′) = anharmonic ratio of the points A′B′C′D′. Hence the Proposition is proved.

Cor. 1.—The six functions formed, as in Def. 1, with the six chords joining the four concyclic points A′B′C′D′, are the six anharmonic ratios of these points.

Cor. 2.—If two △s CAB, C′A′B′ be inscribed in a ⊙, any two sides, viz., one from each △, are divided equianharmonically by the four remaining sides. For, let the sides be AB, A′B′; then the pencils (C . A′BAB′), (C′ . A′BAB′) are equal (*Cor.* 4, Prop. 3).

Prop. 4.—Pascal's Theorem. *If an irregular hexagon be inscribed in a* ⊙, *the intersections of opposite sides, viz.,* 1*st and* 4*th,* 2*nd and* 5*th,* 3*rd and* 6*th, are collinear.*



Let ABCDEFA be the hexagon. The points L, N, M are collinear.

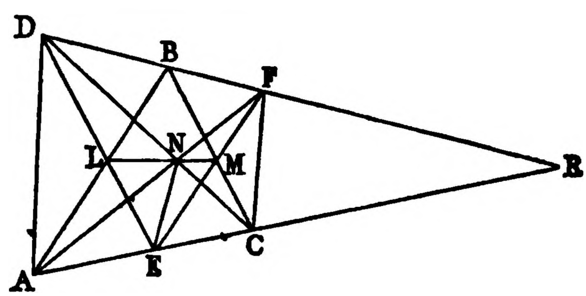
Dem.—Join EN. Then the pencil (N . FMCE) = pencil (C . FBDE), because they

have a common transversal EF (*Cor.* 3, Prop. 3). In like manner, the pencil (A . FBDE) = (N . ALDE); but (A . FBDE) = (C . FBDE) (Prop. 3, *Cor.* 4). Hence the pencils (N . FMCE), (N . ALDE) are equal; and ∴ (*Cor.* 2, Prop. 3) the points L, N, M are collinear. (Q. E. D.)

Cor. 1.—With six points on the circumference of a ⊙, sixty hexagons can be formed. For, starting with any point, say A, we could go from A to one of the remaining points in five ways. Suppose we select B, then we could go from B to a third point in four different ways, and so on; hence it is evident that we could join A to another point, and that again to another, and so on, and finally return to A in $5 \times 4 \times 3 \times 2 \times 1$ different ways. Hence we shall have that number of hexagons; but each is evidently counted twice, and we shall therefore have half the number, that is, sixty distinct hexagons.

Cor. 2.—Pascal's Theorem holds for each of the sixty hexagons.

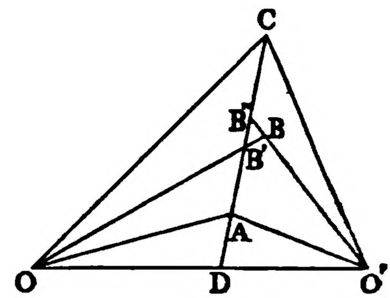
Cor. 3.—Pascal's Theorem holds for six points, which are, three by three, on two lines. Thus, let the



two triads of points be AEC, DBF, and the proof of the Proposition can be applied, word for word, except that the pencil (A . FBDE) is = to the pencil (C . FBDE), for a different reason, viz., they have a common transversal.

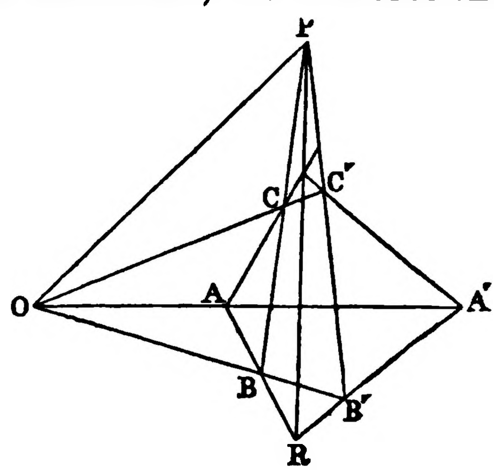
Prop. 5.—*If two equal pencils have a common ray, the intersections of the remaining three homologous pairs of rays are collinear.*

Let the pencils be (O . O′ABC), (O′ . OABC), having the common ray OO′; then, if possible, let the line joining the points A and C intersect the rays OB, O′B in different points B′, B″; then, since the pencils are equal, the anharmonic ratio of the points D, A, B′, C = anharmonic ratio of the points D, A, B″, C, which is impossible. Hence the points A, B, C must be collinear.



Cor. 1. — If A, B, C; A′, B′, C′ be two triads of points on two lines intersecting in O, and if the anharmonic ratio (OABC) = (OA′B′C′), the three lines AA′, BB′, CC′ are concurrent. For, let AA′, BB′, intersect in D; join CD, intersecting OA′ in E; then the anharmonic ratio (OA′B′E) = (OABC) = (OA′B′C′) by hypothesis; ∴ the point E coincides with C′. Hence the Proposition is proved.

Cor. 2—If two △s ABC, A′B′C′ have the lines joining corresponding vertices concurrent, the intersections of corresponding sides must be collinear. For, join P, the point of intersection of the sides BC, B′C′, to O, the centre of perspective; then each of the pencils (A.PCA′B), (A′.PC′AB′) is equal to the pencil (O . PCAB); hence they are = to one another, and they have the ray AA′ common. Hence the intersections of the three corresponding pairs AC, A′C′, AP, A′P′, AB, A′B′, are collinear.



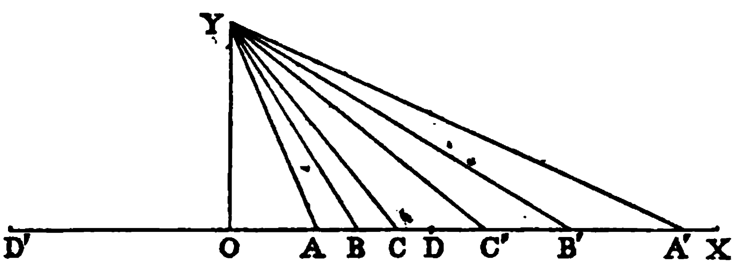
Cor. 3.—If two vertices of a variable △ ABC move on fixed right lines LM, LN, and if the three sides

pass through three fixed collinear points O, P, Q, the locus of the third vertex is a right line.

Let the side AB pass through O, BC through P, CA through Q, and let A′B′C′ be another position of the △; then the two △s AA′Q, BB′Q, have the lines joining their corresponding vertices concurrent; hence the intersections of corresponding sides are collinear. Hence the Proposition is proved.

Prop. 6.—*If on a right line* OX *three pairs of points* A, A′; B, B′; C, C′, *be taken, such that the three rectangles* OA . OA′, OB . OB′, OC . OC′, *are each* = *to a constant, say* k^2, *then the anharmonic ratio of any four of the six points is* = *to the anharmonic ratio of their four conjugates.*

Dem.—Erect OY at right ∠s to OX, and make OY = k; join AY, A′Y, BY, B′Y, CY, C′Y. Now, by hypothesis, OA . OA′ = OY²; ∴ the ⊙ described about the △ AA′Y touches OY at Y; ∴ the ∠ OYA = ∠ OA′Y. In like manner, the ∠ OYB = ∠ OB′Y; hence



the ∠ AYB = ∠ A′YB′; similarly the ∠ BYC = ∠ B′YC′, &c.; ∴ the ∠s of the pencil (Y . ABCC′) = ∠s of the pencil (Y . A′B′C′C); and hence the anharmonic ratio of (Y . ABCC′) = anharmonic ratio of the pencil (Y . A′B′C′C). (Q. E. D.)

Cor. 1.—If the point A moves towards O, the point A′ will move towards infinity.

Cor. 2.—The foregoing Demonstration will hold if some of the pairs of conjugate points be on the production of OX in the negative direction; that is, to the left of OY, while others are to the right, or in the positive direction.

Cor. 3.—If the points A, B, C, &c., be on one side of O, say to the right, their corresponding points A′, B′, C′, &c., may lie on the other side; that is, to the left. In this case the △s AYA′, BYB′, CYC′, &c., are all right angled at Y; and the general Proposition holds for this case also, namely, The anharmonic ratio of any four points is = to the anharmonic ratio of their four conjugates.

Cor. 4.—The anharmonic ratio of any four collinear points is = to the anharmonic ratio of the four points which are inverse to them, with respect to any ⊙ whose centre is in the line of collinearity.

Def.—*When two systems of three points each, such as* A, B, C; A′, B′, C′, *are collinear, and are so related that the anharmonic ratio of any four, which are not two couples of conjugate points, is = to the anharmonic ratio of their four conjugates, the six points are said to be in* involution. *The point* O *conjugate to the point at infinity is called the* centre *of the involution. Again, if we take two points* D, D′, *one at each side of* O, *such that* $OD^2 = OD'^2 = k^2$, *it is evident that each of these points is its own conjugate. Hence they have been called, by* Townsend *and* Chasles, the double points *of the involution.* From these Definitions, the following Propositions are evident:—

(1). *Any pair of homologous points, such as* A, A′, *are harmonic conjugates to the double points* D, D′.

(2). *Three pairs of points which have a common pair of harmonic conjugates form a system in involution.*

(3). *The two double points, and any two pairs of conjugate points, form a system in involution.*

(4). *Any line cutting three coaxal ⊙s is cut in involution.*

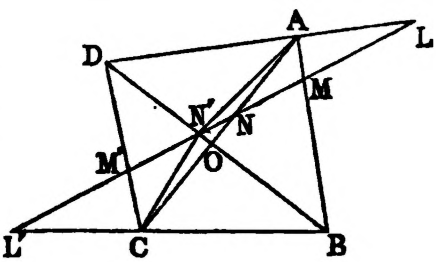
Def.—*If a system of points in involution be joined to any point* P *not on the line of collinearity of the points, the six joining lines will have the anharmonic ratio of the pencil formed by any four rays = to the anharmonic ratio of the pencil formed by their four conjugate rays. Such a pencil is called a* pencil in involution. *The rays*

passing through the double points are called the double rays of the involution.

Prop. 7.—*If four points be collinear, they belong to three systems in involution.*

Dem.—Let the four points be A, B, C, D; upon AB and CD, as diameters, describe ⊙s; then any ⊙ coaxal with these will intersect the line of collinearity of A, B, C, D in a pair of points, which form an involution with the pairs A, B, C, D. Again, describe ⊙s on the segments AD, BC, and ⊙s coaxal with them will give us a second involution. Lastly, the ⊙s described on CA, BD will give us a third system. The central points of these systems will be the points where the radical axes of the coaxal systems intersect the line of collinearity of the points.

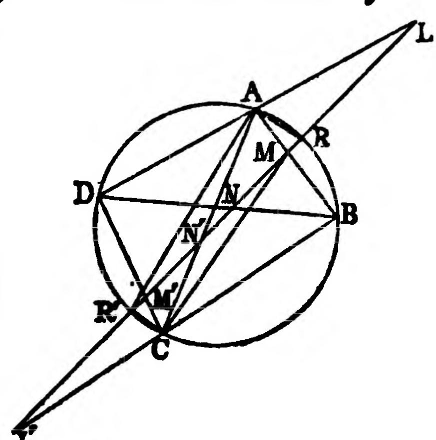
Prop. 8.—The following examples will illustrate the theory of involution:—



(1). *Any right line cutting the sides and diagonals of a quad*[l]. *is cut in involution.*

Dem.—Let ABCD be the quad[l]., LL′ the transversal intersecting the diagonals in the points N, N′. Join AN′, CN′; then the anharmonic ratio of the pencil (A . LMNN′) = (A . DBON′) = (C . DBON′) = (C . M′L′ON′) = (C . L′M′N′N). (Q.E.D.)

(2). *A right line, cutting a ⊙ and the sides of an inscribed quad*[l]., *is cut in involution.*



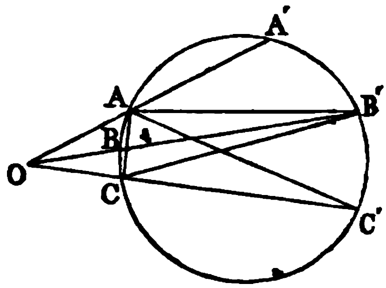
Dem.—Join AR, AR′, CR, CR′; then the anharmonic ratio of the pencil

(A . LRMR′) = (A . DRBR′) = (C . DRBR′) = (C . M′RL′R′) = (C . L′R′M′R).

Cor.—The points N, N′ belong to the involution.

(3). *If three chords of a ⊙ be concurrent, their six points of intersection with the ⊙ are in involution.*

Let AA′, BB′, CC′ be the three chords intersecting in the point O. Join AC, AC′, AB′, CB′; then the anharmonic ratio (A . CA′B′C′) = (B′ . CBAC′) = (B′ . C′ABC). (Q. E. D.)



Cor.—The pencil formed by any six lines from the pairs of homologous points A, A′; B, B; C, C, to any seventh point in the circumference is in involution.

Prop. 9.—*If* O, O′ *be two fixed points on two given lines* OX, O′X′, *and if on* OX *we take any system of points* A, B, C, &c., *and on* O′X′ *a corresponding system* A′, B′, C′, &c., *such that the rectangles* OA . O′A′ = OB . O′B′ = OC . O′C′, &c., = *constant, say* k^2; *then the anharmonic ratio of any four points on* OX = *anharmonic ratio of their four corresponding points on* O′X′.

This is evident by superposition of O′X′ on OX, so that the point O′ will coincide with O (see Prop. 7); then the two ranges on OX will form a system in involution.

Def.—*Two systems of points on two lines, such that the anharmonic ratio of any four points on one line is = to the anharmonic ratio of their four corresponding lines on the other, are said to be* homographic, *and the lines are said to be* homographically *divided. The points* O, O′ *are called the* centres of the systems.

Cor. 1.—The point O on OX is the point corresponding to infinity on O′X′; and the point O′ on O′X′ corresponds to infinity on OX.

Def.—*If the line* O′X′ *be superimposed on* OX, *but so that the point* O′ *will not coincide with* O, *the two systems of points on* OX *divide it homographically, and the points of one system which coincide with their homologous points of the other are called the* double points *of the homography.*

Prop. 10.—*Given three pairs of corresponding points of a line divided homographically, to find the double points.*

Let A, A′; B, B′; C, C′, be the three pairs of corresponding points, and O one of the required double points; then the conditions of the question give us the anharmonic ratio

$$(OABC) = (OA'B'C');$$

$$\therefore \frac{OA \,.\, BC}{OB \,.\, AC} = \frac{OA' \,.\, B'C'}{OB' \,.\, A'C'}.$$

Hence $$\frac{OA \,.\, OB'}{OA' \,.\, OB} = \frac{B'C' \,.\, AC}{BC \,.\, A'C'}$$

= constant, say k^2.

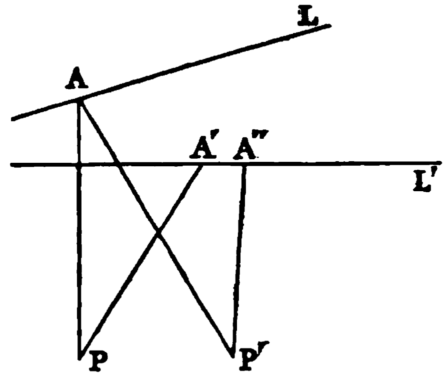
Now OA . OB′, OA′ . OB are the squares of tangents drawn from O to the ⊙s described on the lines AB′ and A′B as diameters; hence the ratio of these tangents is given; but if the ratio of tangents from a variable point to two fixed ⊙s be given, the locus of the point is a ⊙ coaxal with the given ⊙s. Hence the point O is given as one of the points of intersection of a fixed ⊙ with OX, and these are two double points of the homographic system.

If the three pairs of points be on a ⊙, the points of intersection of Pascal's line with the ⊙ will be the double points required. For (see fig., Prop. 5), let D, B, F; A, E, C, be the two triads of points, and let the Pascal's line intersect the ⊙ in the points P and Q; then it is evident that the pencil (A . PDBF) = (D . PAEC).

Cor.—If we invert the ⊙ into a line, or *vice versâ*, the solution of either of the Problems we have given here will give the solution of the other.

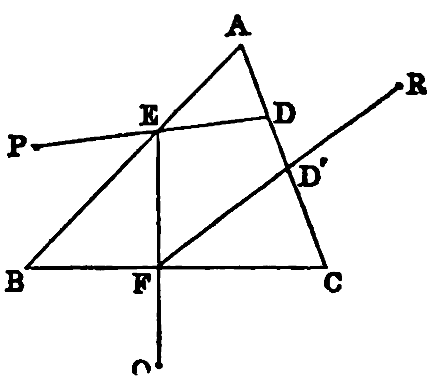
Prop. 11.—*We shall conclude this Section with the solution of a few Problems by means of the double points of homographic division.*

(1). *Being given two right lines* L, L′, *it is required to place between them a line* AA′, *which will subtend given* ∠*s* Ω, Ω′ *at two given points* P, P′.



Solution.—Let us take arbitrarily any point A on L. Join PA, P′A, and make the ∠s APA′, APA″, respectively = to the two given ∠s Ω, Ω′; then, when the point A moves along the line L, the points A′, A″ will form two homographic divisions on the line L′. The two double points of these divisions will give two solutions of the required Problem.

(2). *Being given a polygon of any number of sides, and as many points taken arbitrarily, it is required to inscribe in the polygon another polygon whose sides will pass through the given points.*

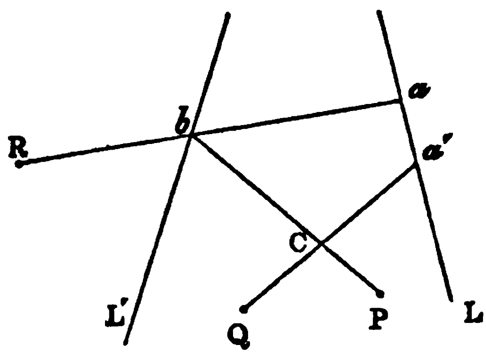


We shall solve this problem for the special case of a △; but it will be seen that the solution is perfectly general.

Let ABC be the given triangle, P, Q, R the given points. Take on AC any arbitrary point D. Join PD, intersecting AB in E; then join EQ, intersecting BC in F; lastly, join FR, intersecting AC in D′; then the two points D, D′ will evidently form two homographic divisions on AC, the two double points of which will be vertices of two △s satisfying the question.

(3). *Being given three points* P, Q, R, *and two lines* L, L′, *it is required to describe a triangle* ABC *having* C *equal to a given* ∠, *the vertices* A *and* B *on the given lines* L, L′, *and the sides passing through the given points.*

Solution.—Through the point R draw any line meeting the two lines L, L′ in the points *a*, *b*. Join *Pb*, and from Q draw Q*a*′ making the required angle C with P*b*; the two points *a*, *a*′ will form two homographic divisions on L, the double points of which will give two solutions of the required question.



(4). *To inscribe in a ⊙ a △ whose sides shall pass through three given points.*

This is evidently solved like the preceding, by taking three false positions, and finding the double points of the two homographic systems of points.

SECTION VII.

Theory of Poles and Polars, and Reciprocation.

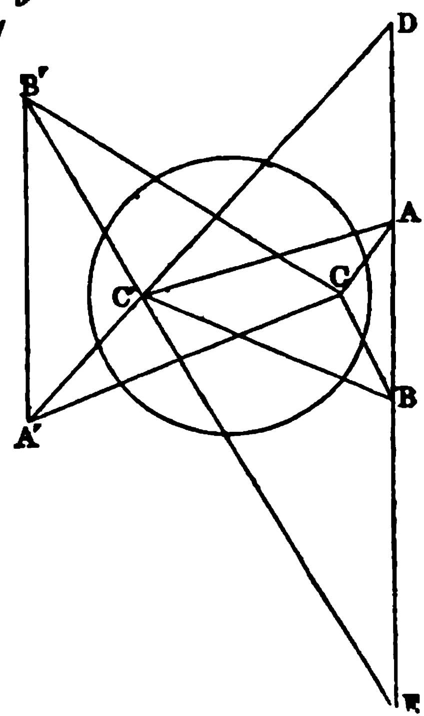
Prop. 1.—*If four points be collinear, their anharmonic ratio is = to the anharmonic ratio of their four polars.*

This Proposition may be proved exactly the same as Proposition 10, Section III. Thus (see fig., Prop. 10, Section III.) the pencil (O . A′B′C′D′) = (P . A′B′C′D′); but the pencil (O . A′B′C′D′) = anharmonic ratio of the four points A, B, C, D, and the pencil (P . A′B′C′D′) consists of the four polars. Hence the Proposition is proved.

The two following Propositions are interesting applications of this Proposition :—

(1). *If two △s be self-conjugate with respect to a ⊙, any two sides are divided equianharmonically by the*

four remaining sides: and any two vertices are subtended equianharmonically by the four remaining vertices.



Let ABC, A′B′C′ be the two self-conjugate △s; it is required to prove that the pencil (C . ABA′B′) = (C′. ABA′B′).

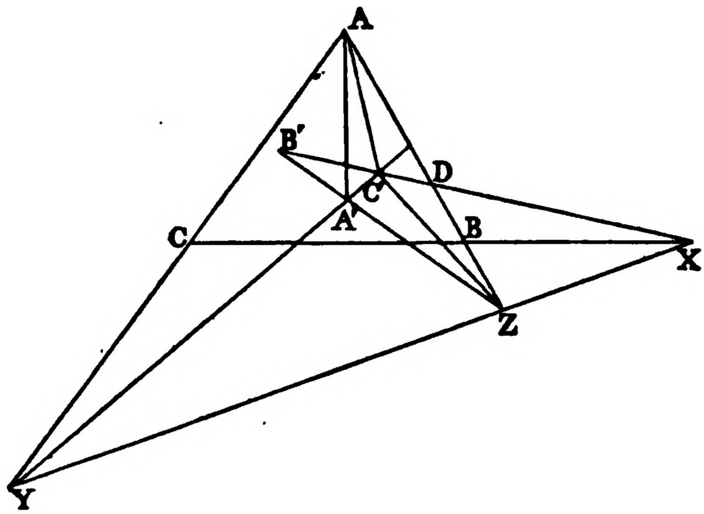
Dem.—Let A′C′, B′C′ meet AB produced in D and E. Join A′C, B′C, AC′, BC′. Now, since A′C′ is the polar of B′, and AB the polar of C, their point of intersection D is the pole of B′C (see *Cor.*, Prop. 25, Section I., Book III.). In like manner, the point E is the pole of A′C; hence the four points B, A, E, D are the poles of the four lines CA, CB, CA′, CB′. Therefore the anharmonic ratio of the four points B, A, E, D is = to the anharmonic ratio of the pencil (C . ABA′B′). Again, the points B, A, E, D are the intersections of the line AB with the pencil (C . ABA′B′); ∴ the pencil (C . ABA′B′) = (C′. ABA′B′).

We have proved the second part of our Proposition, and the first follows from it by the theorem of this Article.

(2). *If two △s be such that the sides of one are the poles of the vertices of the other, they are in perspective.*

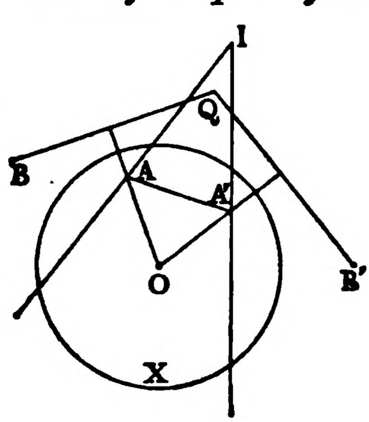
Dem.—Let the three sides of the △ ABC be the polars of the corresponding vertices of the △ A′B′C′, and let the corresponding sides meet in the points X, Y, Z respectively. Now, since AB is the polar of C′, and B′C′ the polar of A, the point D is the pole of AC′ (*Cor.*, Prop. 25, Section I., Book III.). In like manner the point X is the pole of AA′, and the points B′, C′ are, by hypothesis, the poles of the lines AC, AB.

Hence the anharmonic ratio of the points B′, C′, D, X = pencil (A . YZC′A′) = pencil (Z . YAC′A′). Again, the anharmonic ratio B′C′DX = pencil (Z . A′C′AX) = (Z . XAC′A′). Hence (Z . YAC′A′) = (Z . XAC′A′); ∴ the lines XZ, ZY form one right



line ; ∴ the intersection of corresponding sides of the △s are collinear. Hence they are in perspective.

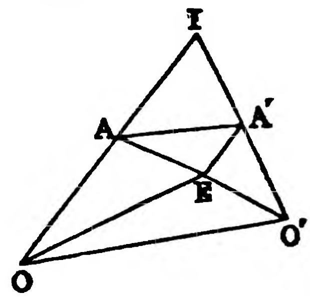
Prop. 2.—*If two variable points* A, A′, *one on each of two lines given in position, subtend an angle of constant magnitude at a given point* O, *the locus of the pole of the line* AA′ *with respect to a given* ⊙ X, *whose centre is* O, *is a* ⊙.



Dem.—Let AI, A′I be the lines given in position, and let B, B′, Q be the poles of the three lines AI, A′I, and AA′ with respect to X; then the points B, B′ are fixed, and the lines BQ, B′Q are the polars of the points A, A′; ∴ the lines OA, OA′ are respectively ⊥ to the lines BQ, B′Q; hence the ∠ BQB′ is the supplement of the ∠ AOA′; ∴ BQB′ is a given ∠, and the

points B, B′ are fixed; ∴ the locus of the point Q is a ⊙.—Q.E.D.

Prop. 3.—*For two homographic systems of points on two lines given in position there exist two points, at each of which the several pairs of corresponding points subtend equal ∠s.*



Dem.—Let A, A′ be two corresponding points on the lines AI. A′I, and let O, O′ be the points on the lines AI, A′I which correspond to the points at infinity on A′I, AI respectively; then (see Prop. 10, Book VI., Section VI.) the rectangle OA . O′A′ = constant, say k^2. Join OO′, and describe the △ OEO′ (see Prop. 15, Book VI., Section I.) having the rectangle OE . O′E of its sides = k^2, and having the difference of its base ∠s = difference of base ∠s of the △ OIO′. Then E, the vertex of this △, will be one of the points required. For it is evident from the construction that OE . O′E = OA . O′A, and that the ∠ AOE = ∠ A′O′E; ∴ the △s AOE, A′O′E are equiangular; ∴ the ∠OAE = ∠ A′EO′; ∴ if the points A, A′ change position, the lines EA, EA′ will revolve in the same direction, and through equal ∠s. Hence the ∠ AEA′ is constant.

In the same manner, another point F can be found on the other side of OO′ such that the ∠ AFA′ is constant.

Cor. 1.—Since the line AA′ subtends a constant ∠ at E, the locus of the pole of AA′ with respect to a ⊙ whose centre is E is a ⊙. Hence the properties of lines joining corresponding points on two lines divided homographically may be inferred from the properties of a system of points on a ⊙.

Cor. 2.—Since when A′ goes to infinity A coincides with O, then OA is one of the lines joining corresponding points. And so in like manner is O′A′, and the poles of these lines will be points on the ⊙ which is the locus of the pole of AA′.

Cor. 3.—The locus of the foot of the $\perp$ from E on the line AA′ is a $\odot$, namely, the inverse of the $\odot$ which is the locus of the pole of AA′.

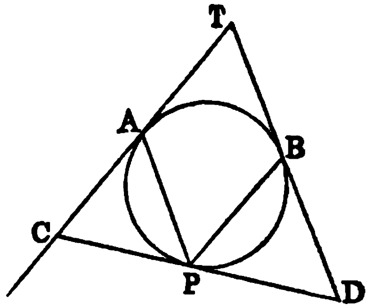
Cor. 4.—If two lines be divided homographically, any four lines joining corresponding points are divided equianharmonically by all the remaining lines joining corresponding points. This follows from the fact that any four points on a $\odot$ are subtended equianharmonically by all the remaining points of the $\odot$.

Prop. 4.—*If any figure* A *be given, by taking the pole of every line, and the polar of every point in it with respect to any arbitrary* $\odot$ X, *we can construct a new figure* B, *which is called the reciprocal of* A *with respect to* X. *Thus we see that to any system of collinear points or concurrent lines of* A *there will correspond a system of concurrent lines or collinear points of* B; *and to any pair of lines divided homographically in* A *there will correspond in* B *two homographic pencils of lines. Lastly, the* $\angle$ *which any two points of* A *subtend at the centre of the reciprocating* $\odot$ *is* = *to the* $\angle$ *made by their polars in* B.

Hence it is evident that from theorems which hold for A we can get other theorems which are true for B. This method, which is called reciprocation, is due to Poncelet, and is one of the most important known to Geometers.

We give a few Theorems proved by this method:—

(1). *Any two fixed tangents to a* $\odot$ *are cut homographically by any variable tangent.*



Dem.—Let AT, BT be the two fixed tangents touching the $\odot$ at the fixed points A and B, and CD a variable tangent touching at P. Join AP, BP. Now AP is the polar of C, and BP the polar of D; and if the point P take four different positions, the point C will take four different positions, and so will the point D, and the anharmonic

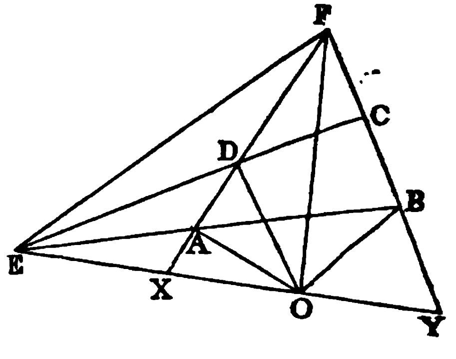
ratio of the four positions of C = anharmonic ratio of the pencil from A to the four positions of P (Prop. 1). Similarly, the anharmonic ratio of the four positions of D = anharmonic ratio of the pencil from B to the four positions of P; but the pencil from A = pencil from B; ∴ the anharmonic ratio of the four positions of C = anharmonic ratio of the four positions of D.

(2). *Any four fixed tangents to a ⊙ are cut by any fifth variable tangent in four points whose anharmonic ratio is constant.*

Dem.—The lines joining the point of contact of the variable tangent to the points of contact of the fixed tangents are the polars of the points of intersection of the variable tangent with the fixed tangents; but the anharmonic ratio of the pencil of four lines from a variable point to four fixed points on a ⊙ is constant; hence the anharmonic ratio of their four poles—that is, of the four points in which the variable tangent cuts the fixed tangent—is constant.

(3). *Lines drawn from any variable point in the plane of a quadrilateral to the six points of intersection of its four sides form a pencil in involution.*

This Proposition is evidently the reciprocal of (1), Prop. 9, Section VI. The following is a direct proof: Let ABCD be the quadrilateral, and let its opposite sides meet in the points E and F, and O the point in the plane of the quadrilateral; the pencil from O to the points A, B, C, D, E, F is in involution.



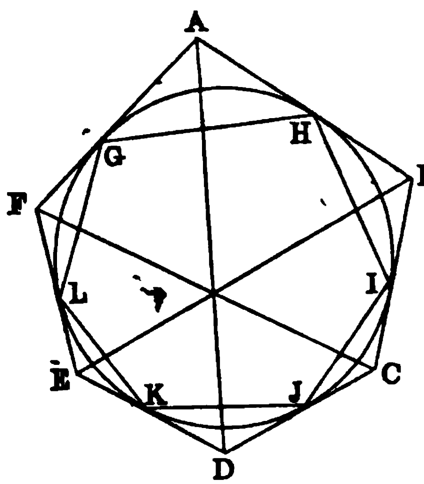
Dem.—Join OE, cutting the sides AD, BC in X and Y. Join EF. Now, the pencil (O . XADF) = (E . XADF) = (E . YBCF) = (O . YBCF) = (O . XBCF); ∴ (O . EADF) = (O . EBCF). Hence the pencil is in involution.

(4). *If two vertices of a △ move on fixed lines, while the three sides pass through three collinear points, the locus of the third vertex is a right line.* Hence, reciprocally, *If two sides of a △ pass through fixed points, while the vertices move on three concurrent lines, the third side will pass through a fixed point.*

(5). *To describe a △ about a ⊙, so that its three vertices may be on three given lines.* This is solved by inscribing in the ⊙ a △ whose three sides shall pass through the poles of the three given lines, and drawing tangents at the angular points of the inscribed △.

(6). BRIANCHON'S THEOREM.—*If an irregular hexagon be described about a ⊙, the three lines joining the opposite angular points are concurrent.*

This is the reciprocal of Pascal's Theorem: we prove it as follows:—



Let ABCDEF be the circumscribed hexagon; the three diagonals AD, BE, CF are concurrent. For, let the points of contact be G, H, I, J, K, L. Now, since A is the pole of GH, and D the pole of JK, the line AD is the polar of the point of intersection of the opposite sides GH and JK of the inscribed hexagon. In like manner, BE is the polar of the point of intersection of the lines HI, KL, and CF the polar of the point of intersection of IJ and LG; but the intersections of the three pairs of opposite sides of the inscribed hexagon, viz., GH, JK; HI, KL; IJ, LG, are, by Pascal's theorem, collinear; ∴ their three polars AD, BE, CF, are concurrent.

(7). *If two lines be divided homographically, two lines joining homologous points can be drawn, each of which passes through a given point.*

For, if AA′ (see fig., Prop. 3) pass through a given point P, join EP, and let fall a ⊥ EG on AA′; then (*Cor.* 2, Prop. 3), the locus of the point G is a ⊙; and since EGP is a right ∠, the ⊙ described on EP as diameter passes through G; hence G is the point of intersection of two given ⊙s; and since two ⊙s intersect in two points, we see that two lines joining homographic points can be formed, each passing through P. Now, if we reciprocate the whole diagram with respect to a ⊙ whose centre is P, the reciprocals of the points A, A′ will be parallel lines. Hence we have the following theorem in a system of two homographic pencils of rays:—*There exist two pairs of homologous rays which are parallel to each other.*

Cor.—There are two directions in which transversals can be drawn, intersecting two homographic pencils of rays so as to be divided proportionally, namely, parallel to the pairs of homologous rays which are parallel.

(8). If we reciprocate Prop. 3 we have the following theorem:—*Being given a fixed point, namely, the centre of the ⊙ of reciprocation and two homographic pencils of rays, two lines can be found* (*the polars of the points* E *and* F *in* Prop. 3), *so that the portions intercepted on each by homologous rays of the pencils will subtend an ∠ of constant magnitude at the given point.*

SECTION VIII.

Miscellaneous Exercises.

1. The lines from the angles of a △ to the points of contact of any ⊙ touching the three sides are concurrent.

2. Three lines being given in position, to find a point in one of them, such that the sum of two lines drawn from it making given ∠s with the other two may be given.

3. From a given point in the diameter of a semicircle produced to draw a line cutting the semicircle, so that the lines may have a given ratio which join the points of intersection to the extremities of the diameter.

4. The internal and external bisectors of the vertical angle of a △ meet the base in points which are harmonic conjugates to the extremities.

5. The rectangle contained by the sides of a △ is greater than the square of the internal bisector of the vertical ∠ by the rectangle contained by the segments of the base.

6. State the corresponding theorem for the external bisector.

7. Given the base and the vertical angle of a △, find the following loci:—

(1). Of the intersection of ⊥s.

(2). Of the centre of any ⊙ touching the three sides.

(3). Of the centre of the circumscribed ⊙.

(4). Of the intersection of bisectors of sides.

8. If a variable ⊙ touch two fixed ⊙s, the tangents drawn to it from the limiting points have a constant ratio.

9. The ⊥ from the right ∠ on the hypotenuse of a right-angled △ is a harmonic mean between the segments of the hypotenuse made by the point of contact of the inscribed ⊙.

10. If a line be cut harmonically by two ⊙s, the locus of the foot of the ⊥, let fall on it from either centre, is a ⊙, and it cuts any two positions of itself homographically (see Prop. 3, *Cor.* 2, Section VII.).

11. Through a given point to draw a line, cutting the sides of a given △ in three points, such that the anharmonic ratio of the system, consisting of the given point and the points of section, may be given.

12. If squares be described on the sides of a △ and their centres joined, the area of the △ so formed exceeds the area of the given △ by $\frac{1}{8}$th part of the sum of the squares.

13. The locus of the centre of a ⊙ bisecting the circumferences of two fixed ⊙s is a right line.

14. Divide a given semicircle into two parts by a ⊥ to the diameter, so that the diameters of the ⊙s described in them may be in a given ratio.

15. The side of the square inscribed in a △ is half the harmonic mean between the base and ⊥.

16. The ⊙s described on the three diagonals of a quad[l]. are coaxal.

17. If X, X′ be the points where the bisectors of the $\angle$ A of a $\triangle$ and of its supplement meet the side BC, and if Y, Y′; Z, Z′, be points similarly determined on the sides CA, AB: then

$$\frac{1}{XX'} + \frac{1}{YY'} + \frac{1}{ZZ'} = 0;$$

and
$$\frac{a^2}{XX'} + \frac{b^2}{YY'} + \frac{c^2}{ZZ'} = 0.$$

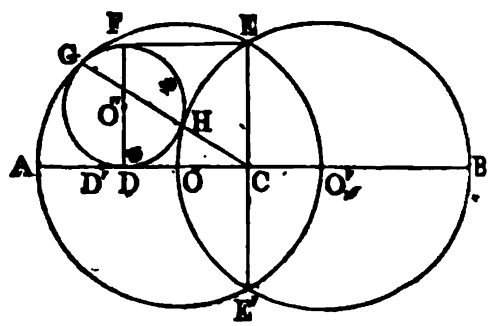
18. Prove Ptolemy's theorem, and its converse, by inversion.

19. A line of given length slides between two fixed lines; find the locus of the intersection of the $\perp$s to the fixed lines from the extremities of the sliding line, and of the $\perp$s on the fixed lines from the extremities of the sliding line.

20. If from a variable point P $\perp$s be drawn to three sides of a $\triangle$; then, if the area of the $\triangle$ formed by joining the feet of these $\perp$s be given, the locus of P is a $\odot$.

21. If a variable $\odot$ touch two fixed $\odot$s, its radius varies as the square of the tangent drawn to it from either limiting point.

22. If two $\odot$s, whose centres are O, O′, intersect, as in Euclid (I. 1), and OO′ be joined, and produced to A, and a $\odot$ GDH be described, touching the $\odot$s whose centres are O, O′, and also touching the line AO; then, if we draw the radical axis EE′ of the $\odot$s, intersecting OO′ in C, and the diameter DF of the $\odot$ GHD, and join EF, the figure CDFE is a square.



Dem.—The line joining the points of contact G and H will pass through C, the internal centre of similitude of the $\odot$s O, O′; therefore CG . CH = CE², but CD² = CG . CH; $\therefore$ CD = CE.

Again, let O″ be the centre of GDH, and D′ the middle point of AO; then the $\odot$ whose centre is D′ and radius D′A touches the $\odot$s O, O′; hence (by Theorem 7, Section V.) the $\perp$ from O″ on EE′ : O″D :: CD′ : D′A; that is, in the ratio of 2 : 1. Hence the Proposition is proved.

23. If a quad¹. be circumscribed to a $\odot$, the centre and the middle points of the diagonals are collinear.

24. If one diagonal of a quad¹. inscribed in a $\odot$ be bisected by the other, the square of the latter = ½ sum of squares of sides.

25. If a $\triangle$ given in species moves with its vertices on three fixed lines, it marks off proportional parts on these lines.

26. Through the point of intersection of two $\odot$s draw a line so that the sum or the difference of the squares of the chords of the $\odot$s shall be given.

27. If two $\odot$s touch at A, and BC be any chord of one touch the other; then the sum or difference of the chords AB, AC bears to the chord BC a constant ratio. Distinguish the two cases.

28. If ABC be a $\triangle$ inscribed in a $\odot$, and if a $\parallel$ to AC through the pole of AB meet BC in D, then AD is = CD.

29. The centres of the four $\odot$s circumscribed about the $\triangle$s, formed by four right lines, are concyclic.

30. Through a given point draw two transversals which shall intercept given lengths on two given lines.

31. If a variable line meet four fixed lines in points whose anharmonic ratio is constant, it cuts these four lines homographically.

32. Given the $\perp$ CD to the diameter AB of a semicircle, it is required to draw through A a chord, cutting CD in E and the semicircle in F, such that the ratio of CE : EF may be given.

33. Draw in the last construction the line CEF so that the quad[l]. CDEF may be a maximum.

34. The $\odot$ described through the centres of the three escribed $\odot$s of a plane $\triangle$, and the circumscribed $\odot$ of the same $\triangle$, will have the centre of the inscribed $\odot$ of the $\triangle$ for one of their centres of similitude.

35. The $\odot$s on the diagonals of a complete quad[l]. cut orthogonally the $\odot$ described about the $\triangle$ formed by the three diagonals.

36. When the three $\perp$s from the vertices of one $\triangle$ on the sides of another are concurrent, the three corresponding $\perp$s from the vertices of the latter, on the sides of the former, are concurrent.

37. If a $\odot$ be inscribed in a quadrant of a $\odot$, and a second $\odot$ be described touching the $\odot$, the quadrant, and radius of quadrant, and a $\perp$ be let fall from the centre of the second $\odot$ on the line passing through the centres of the first $\odot$ and of the quadrant, then the $\triangle$ whose angular points are the foot of the $\perp$, the centre of the quadrant, and the centre of the second $\odot$, has its sides in arithmetical progression.

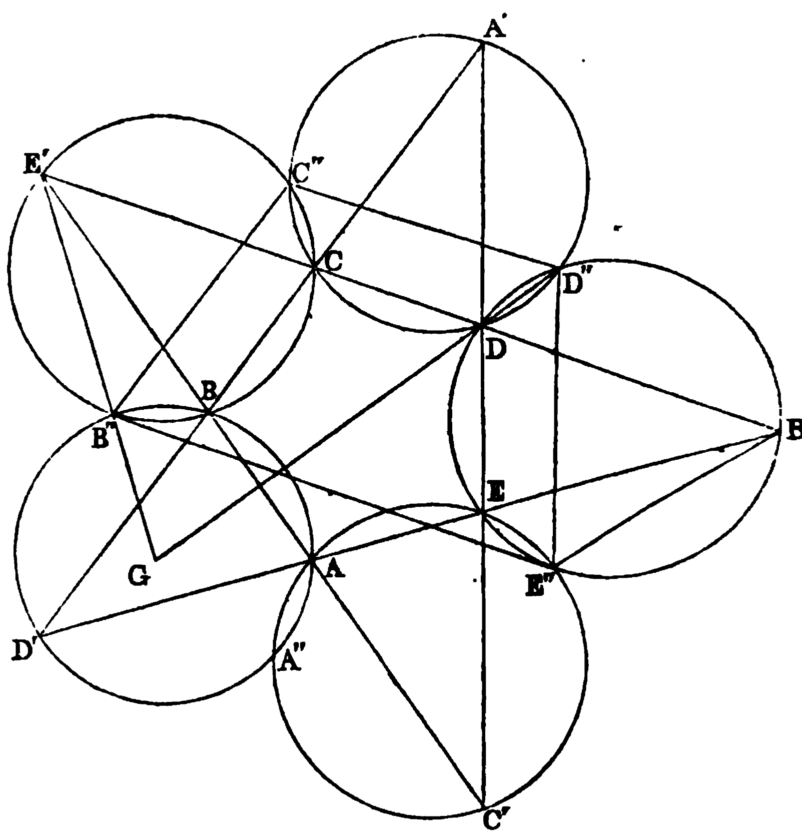
38. In the last Proposition, the $\perp$s let fall from the centre of the second $\odot$ on the radii of the quadrants are in the ratio of 1 : 7.

39. When three ⊙s of a coaxal system touch the three sides of a △ at three points, which are either collinear or concurrently connectant with the opposite vertices, their three centres form, with those of the three ⊙s of the system which pass through the vertices of the △, a system of six points in involution.

40. If two ⊙s be so placed that a quad[l]. may be inscribed in one and circumscribed to the other, the diagonals of the quad[l]. intersect in one of the limiting points.

41. If from a fixed point ⊥s be let fall on two conjugate rays of a pencil in involution, the feet of the ⊥s are collinear with a fixed point.

42. Miquel's Theorem.—If the five sides of any pentagon ABCDE be produced, forming five △s external to the pentagon, the ⊙s described about these △s intersect in five points A″, B″, C″, D″, E″, which are concyclic.



Dem.—Join E″B′, E″D″, D″C″, C″B″, C″C; join also D″D and E′B″, and let them produced meet in G. Now, consider the △ AB′E′, it is evident the ⊙ described about it (*Cor.* 3, Prop. 12, Book III.) will pass through the points E″, B″; hence the fom

points E″, B′, E′, B″ are concyclic; ∴ the ∠ GB″E″ = ∠ E′B′E″, but E′B′E″ = GD″E″; ∴ ∠ GB″E″ = ∠ GD″E″. Hence the ⊙ through the points B″, D″, E″ passes through G.

Again, since the figure CDD″C″ is a quad^l. in a ⊙, the ∠ GDE′ = ∠ D″C″C, and the ∠ GE′D = ∠ B″C″C (III. 21); ∴ ∠ B″C″D″ = ∠ GDE′ + ∠ GE′D; to each add ∠ E′GD, and we see that the figure GD″C″B″ is a quad^l. in a ⊙; hence the ⊙ through the points B″, D″, E″ passes through C″. In like manner it passes through A″. Hence the five points A″, B″, C″, D″, E″ are concyclic.

43. If the product of the tangents, from a variable point P to two given ⊙s, has a given ratio to the square of the tangent from P to a third given ⊙, coaxal with the former, the locus of P is a ⊙ of the same system.

44. Through the vertices of any △ are drawn any three parallel lines, and through each vertex a line is drawn, making the same ∠ with one of the adjacent sides which the parallel makes with the other, these three lines are concurrent. Required the locus of the point in which they meet.

45. If from any point in a given line two tangents be drawn to a given ⊙, X, and if a ⊙, Y, be described touching X and the two tangents, the envelope of the polar of the centre of Y with respect to X is a ⊙.

46. The extremities of a variable chord XY of a given ⊙ are joined to the extremities of a fixed chord AB; then, if m AX . AY + n BX . BY be given, the envelope of XY is a ⊙.

47. If A, A′ be conjugate points of a system in involution, and if AQ, A′Q be ⊥ to the lines joining A, A′ to any fixed point P, it is required to find the locus of Q.

48. If a, a', b, b', c, c', be three pairs of conjugate points of a system in involution; then,

(1). $$ab' \, . \, bc' \, . \, ca' = - a'b \, . \, b'c \, . \, c'a.$$

(2). $$ab' \, . \, bc \, . \, c'a' = - a'b \, . \, b'c' \, . \, ca.$$

(3). $$\frac{ab \, . \, ab'}{ac \, . \, ac'} = \frac{a'b \, . \, a'b'}{a'c \, . \, a'c'}.$$

49. Construct a right-angled △, being given the sum of the base and hypotenuse, and the sum of the base and perpendicular.

50. Given the perimeter of a right-angled △ whose sides are in arithmetical progression: construct it.

51. Given a point in the side of a △; inscribe in it another △ similar to a given △, and having one ∠ at the given point.

52. Given a point D in the base AB produced of a given △ ABC; draw a line EF through D cutting the sides so that the area of the △ EFC may be given.

53. Construct a △ whose three ∠s shall be on given ⊙s, and whose sides shall pass through three of their centres of similitude.

54. From a given point O three lines OA, OB, OC are drawn to a given line ABC; prove that if the radii of the ⊙s inscribed in OAB, OBC are given, the radius of the ⊙ inscribed in OAC will be determined.

55. Equal portions OA, OB are taken on the sides of a given right ∠ AOB, the point A is joined to a fixed point C, and a ⊥ let fall on AC from B, the locus of the foot of this ⊥ is a ⊙.

56. If a segment AB of a given line be cut in a given anharmonic ratio in two variable points X, X′, then the anharmonic ratio of any four positions of X will be = to the anharmonic ratio of the four corresponding positions of X′.

57. If a variable △ inscribed in a ⊙, X, whose radius is R, has two of its sides touching another ⊙, Y, whose radius is r, and whose centre is distant from the centre of X by δ; then the distance of the centre of the ⊙ coaxal with X and Y, which is the envelope of the third side of the △ from the centre of X,

$$= r^2 \delta \div \frac{(R^2 - \delta^2)^2}{4R^2}.$$

58. In the same case the radius of the ⊙ which is the envelope of the third side is

$$\frac{r^2 (R - \rho) - R\rho^2}{\rho^2};$$

where $$\rho = \frac{R^2 - \delta^2}{2R}.$$

59. If two tangents be drawn to a ⊙, the points where any third tangent is cut by these will be harmonic conjugates to the point of contact and the point where it is cut by the chord of contact.

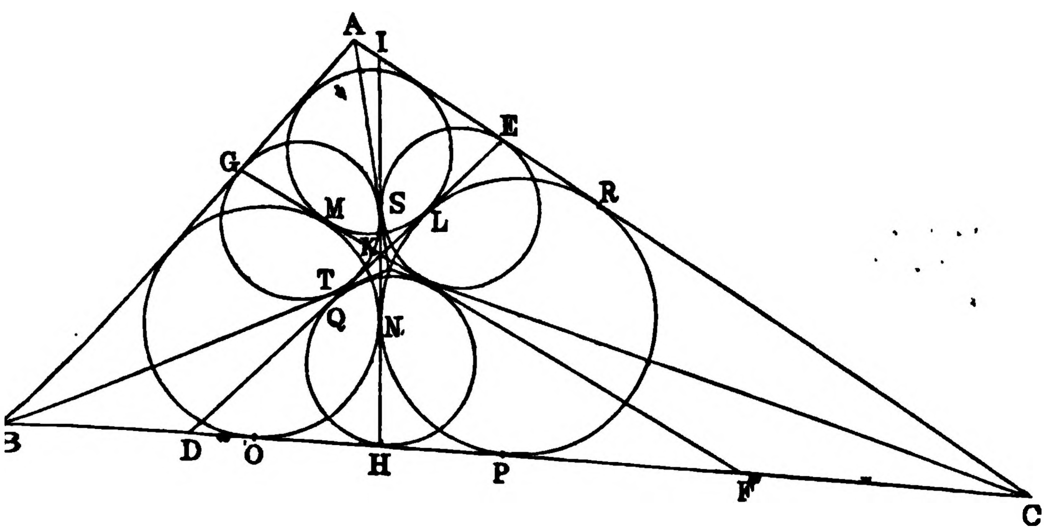
60. If two points be inverse to each other with respect to any ⊙, then the inverses of these will be inverse to each other with respect to the inverse of the ⊙. Hence it follows that if two figures be inverse to each other with respect to any ⊙, their inverses will be inverse to each other with respect to the inverse of the ⊙.

61. MALFATTI'S PROBLEM.—To inscribe in a △ three ⊙s which touch each other, and each of which touches two sides of the △.

Analysis.—Let L, M, N be the points of contact of three ⊙s which touch one another, and each touch two sides of the △ ABC; draw the common tangents DE, FG, HI to these ⊙s at their points of contact L, M, N; then, since these lines are the radical axes of the ⊙s taken in pairs, they are concurrent; let them meet in K.

Now, it is evident that FH - HD = FO - DP = FM - DL = FK - DK. Hence H is the point of contact with FD of the ⊙ described in the △ FKD. In like manner, E and G are the points of contact of ⊙s which touch the triads of lines IK, KF, AC; and IK, DK, AB, respectively.

Again, HN = HP = QL, and NS = ER = EL; ∴ HS = EQ; ∴ (see 6, Prop. 1, Section V.) the tangents at E and H to the ⊙s



ES and HQ meet on their ⊙ of similitude; ∴ C is a point on the ⊙ of similitude of the ⊙s ES and HQ; and therefore these ⊙s subtend equal ∠ s at C. Also, three common tangents of the ⊙s HQ, ES, PNR, viz., QL, SN, KF, are concurrent; ∴ (see Ex. 48, Section II., Book III.) C must be the point of concurrence of three other common tangents to the same ⊙s. Hence the second transverse common tangent to HQ and ES must pass through C; and since C is a point on their ⊙ of similitude, this transverse common tangent must bisect the ∠ ACB. In like manner it is proved that the bisectors of A and B are transverse common tangents to the ⊙s ES and GT, and to HQ and GT, respectively. Hence, we have the following elegant construction:—Let V be the point of concurrence of the three bisectors of the ∠ s of the △ ABC. In the △s VAB, VBC, VCA, describe three ⊙s: these ⊙s

will evidently, taken in pairs, have VB, VC, VA as transverse common tangents; then to the same pairs of ⊙s draw the three other transverse tangents; these will be respectively ED, GF, HI; and the ⊙s described touching the triads of lines AB, AC, ED; AB, BC, GF; AB, BC, HI, will be the required ⊙s.

This construction is due to Steiner, and the foregoing simple and elementary proof to Dr. Hart (see *Quarterly Journal*, vol. i. p. 219).

62. If a transversal passing through a fixed point O cut any number of fixed lines in the points A, B, C, &c., and if P be a point such that

$$\frac{1}{OP} = \frac{1}{OA} + \frac{1}{OB} + \frac{1}{OC} + \&c.,$$

the locus of P is a right line.

63. The sum of the squares of the radii of the four ⊙s, cutting orthogonally the inscribed and escribed ⊙s of a plane △, taken three by three, is equal to the square of the diameter of the circumscribed ⊙.

64. Describe through two given points a ⊙ cutting a given arc of a given ⊙ in a given anharmonic ratio.

65. All ⊙s which cut three fixed ⊙s at equal ∠s form a coaxal system.

66. Being given five points and a line, find a point on the line, so that the pencil formed by joining it to the five given points shall form an involution with the line itself.

67. If a quad^l. be inscribed in a ⊙, the ⊙ described on the third diagonal as diameter will be the ⊙ of similitude of the ⊙s described on the other diagonals as diameters.

68. If ABC be any △, B′C′ a line drawn ∥ to the base BC; then, if O, O′ be the escribed ⊙s to ABC, opposite the ∠s B and C respectively, O_1 the inscribed ⊙ of AB′C′, and O'_1 the escribed ⊙ opposite the ∠ A; then, besides the lines AB, AC, which are common tangents, O, O′, O_1, O'_1, are all touched by two other ⊙s.

69. When two ⊙s intersect orthogonally, the locus of the point whence four tangents can be drawn to the ⊙s, and forming a harmonic pencil, consists of two lines, viz., the polars of the centre of similitude of the two ⊙s.

70. If two lines be divided homographically in the two systems of points a, b, c, &c., a', b', c', &c., then the locus of the points of intersection of ab', $a'b$, ac', $a'c$, ad', $a'd$, &c., is a right line.

71. Being given two homographic pencils, if through the point of intersection of two corresponding rays we draw two transversals, which meet the two pencils in two series of points, the lines joining corresponding point of intersection are concurrent.

72. Inscribe a △ in a ⊙ having two sides passing through two given points, and the third ‖ to a given line.

73. If two △s be described about a ⊙, the six angular points are such that any four are subtended equianharmonically by the other two.

74. Given four points A, B, C, D on a given line, find two other points X, Y so that the anharmonic ratios (ABXY), (CDXY) may be given.

75. If two quad[ls]. have the same diagonals, the eight points of intersection of their sides are such that any four are subtended equianharmonically by the other four.

76. Given three rays A, B, C, find three other rays X, Y, Z through the same vertex O, so that the anharmonic ratios of the pencils (O . ABXY), (O . BCYZ), (O . CAZX), may be given.

77. If a △ given in species slide with its three vertices on three given ⊙s, the vertices divide the ⊙s homographically.

78. Find the locus of the centre of a ⊙, being given that the polar of a given point A passes through a given point B, and the polar of another given point C passes through a given point D.

79. If a △ be self-conjugate with respect to a given ⊙, the ⊙ described about the △ is orthogonal to another given ⊙.

80. The ⊙s self-conjngate to the △s formed by four lines are coaxal.

81. The pencil formed by lines ‖ to the sides and diagonals of a quad[l]. is involution.

82. If four ⊙s be co-orthogonal, that is, have a common orthogonal ⊙, their radical axes form a pencil in involution.

83. In a given ⊙ to inscribe a △ whose sides shall divide in a given anharmonic ratio given arcs of the ⊙.

84. When four ⊙s have a common point of intersection, their six radical axes form a pencil in involution.

85. The pencil formed by drawing tangents from any point in their radical axis to two ⊙ s, and drawing two lines to their centres of similitude, is in involution.

*86. If the opposite ∠s of a quad[l]. be = to one or three right ∠s, then the sum of the squares of the rectangles contained by the opposite sides is = to the square of the rectangle contained by the diagonals.

* This Theorem is due to Bellavitis. See his *Methode des Equipollences*.

87. A straight line moves with its extremities P, Q in the sides of a $\triangle$ OAB, so that the rectangle OP . OQ = AP . BQ; the points P, Q divide the lines OA, BO proportionally.

88. The line joining the intersection of the $\perp$s of a $\triangle$ to the centre of a circumscribed $\odot$ is $\perp$ to the axis of perspective of the given $\triangle$, and the $\triangle$ formed by joining the feet of the $\perp$s.

89. If two $\odot$s whose radii are R, R′, and the distance of whose centres is δ, be such that a hexagon can be inscribed in one and circumscribed to the other; then

$$\frac{1}{(R^2-\delta^2)^2+4R'^2R\delta}+\frac{1}{(R^2-\delta^2)^2-4R'^2R\delta}$$
$$=\frac{1}{2R'^2(R^2+\delta^2)-(R^2-\delta^2)^2}.$$

90. In the same case, if an octagon be inscribed in one and circumscribed to the other,

$$\left\{\frac{1}{(R^2-\delta^2)^2+4R'^2R\delta}\right\}^2+\left\{\frac{1}{(R^2-\delta^2)^2-4R'^2R\delta}\right\}^2$$
$$=\left\{\frac{1}{2R'^2(R^2+\delta^2)-(R^2-\delta)^2}\right\}^2.$$

91. If a variable $\odot$ touch two fixed $\odot$s, the polar of its centre with respect to either of the fixed $\odot$s touches a fixed $\odot$.

92. If a $\odot$ touch three $\odot$s, the polar of its centre, with respect to any of the three $\odot$s, is a common tangent to two $\odot$s.

*93. Prove that the Problem to inscribe a quad^l., whose perimeter is a minimum in another quad^l., is indeterminate or impossible, according as the given quad^l. has the sum of its opposite $\angle$s = or not = to two right $\angle$s.

94. If a quad^l. be inscribed in a $\odot$, the lines joining the feet of the $\perp$s, let fall on its sides from the point of intersection of its diagonals, will form an inscribed quad^l. Q of minimum perimeter, and an indefinite number of other quad^{ls}. may be inscribed whose sides are respectively $\parallel$ to the sides of Q, the perimeter of each of them being = to the perimeter of Q.

95. The perimeter of Q is = to the rectangle contained by the diagonals of the original quad^l. divided by the radius of the circumscribed $\odot$.

* The Theorems 93–96 have been communicated to the author by W. S. M'CAY, F.T.C.D.

96. Being given four lines forming four $\triangle$s, the sixteen centres of the inscribed and escribed $\odot$s to these $\triangle$s lie four by four on four coaxal $\odot$s.

97. If the base of a $\triangle$ be given, both in magnitude and position, and the ratio of the sum of the squares of the sides to the area, the locus of the vertex is a $\odot$.

98. If a line of constant length slide between two fixed lines, the locus of the centre of instantaneous rotation is a $\odot$.

99. If two sides of a $\triangle$ given in species and magnitude slide along two fixed $\odot$s, the envelope of the third side is a $\odot$. (Bobillier).

100. If the lengths of the sides of the $\triangle$ in Ex. 99 be denoted by a, b, c, and the radii of the three $\odot$s by α, β, γ; then $a\alpha \pm b\beta \pm c\gamma =$ twice the area of the $\triangle$, the sign + or − being used according as the $\odot$s touch the sides of the $\triangle$ internally or externally.

101. If five quad^ls^. be formed from five lines by omitting each in succession, the lines of collinearity of the middle points of their diagonals are concurrent. (H. Fox Talbot).

THE END.